Do Sunspots Matter?

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Can extrinsic uncertainty ("animal spirits," "market psychology," "sunspots,". . .) play a significant role in rational expectations equilibrium models? We establish that extrinsic uncertainty cannot matter in the static Arrow-Debreu economy with complete markets. But we also establish that extrinsic uncertainty can matter in the overlapping-generations economy with complete markets but where market participation is limited to those consumers alive when the markets are open. Equilibrium allocations in which extrinsic uncertainty plays no role are Pareto optimal in the traditional sense. Equilibrium allocations in which extrinsic uncertainty does play a role are Pareto optimal in a (weaker) sense which is appropriate to dynamic analysis.

I. Introduction

What is the best strategy for playing the stock market? Should one concentrate on "fundamentals" or should one instead focus on the "psychology of the market"? These are interesting questions for those who would like to get rich on Wall Street. They are also interesting for macroeconomists.

For Keynes and many Keynesians, the volatility of investment is at least in part based on the volatility of market psychology or the animal spirits of capitalists, or, more generally, extrinsic uncertainty. This alleged unreliability of the intertemporal allocation of resources under laissez-faire capitalism is an important basis of the Keynesian

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claim for the desirability of active fiscal and monetary policies. On the other hand, devotees of the rational expectations school of macroeconomics hold that fundamentals play the determining role in the allocation of resources. Therefore, they claim that if the environment is stable (in particular, if the government does not behave capriciously), then private investment will also be appropriately stable.

In this paper, we investigate whether or not rational expectations equilibria are immune to influence from extrinsic uncertainty. That is, Can self-fulfilling prophecies have a real impact? Or, in short, Do sunspots[1] matter?[2]

We consider a simple exchange economy with a finite time horizon, two commodities, two states of nature, and a finite number of consumers. In Section II, consumers are introduced. They are assumed to have von Neumann–Morgenstern utility functions and commodity endowments which are independent of the state of nature. Since uncertainty does not then have any effect on the fundamentals of the economy, it is purely extrinsic uncertainty; we refer to it simply as sunspot activity. Throughout most of the paper, we adopt a very strong version of the rational expectations hypothesis: Consumers share the same beliefs about sunspot activity. This allows the interpretation that subjective probabilities are equal to objective probabilities.

The usual extensions of general equilibrium theory fail to represent an essential and significant aspect of actual intertemporal economies: The trading process must take place in time. Individuals may be able to make complicated trades involving future deliveries contingent upon the revealed state of nature, but these trades must be struck when each of the parties is alive. Markets in all commodities may be complete, but consumers are restricted from participating in markets which meet (say) before they are born.

We assume that markets are complete. In Section III, we describe

[1] Throughout this essay, "sunspots" is meant to represent extrinsic uncertainty, that is, random phenomena that do not affect tastes, endowments, or production possibilities. Of course, as Jevons noted, real-world sunspots may very well provide *intrinsic* uncertainty to the economy, affecting, e.g., agricultural production possibilities. Here we are interested only in our highly stylized version of sunspot activity.

[2] We have been worrying about sunspots for some time. A theoretical example of a monetary overlapping-generations economy in which sunspots matter is reported in Shell (1977). Our colleague Costas Azariadis (1981) has presented similar results in a more familiar macroeconomic setting. These examples, however, are based on very complicated dynamic models which do not permit us to separate the respective roles of generational overlap, the infinite time horizon, and government debt. In the present paper, we focus on the implications of generational overlap in a very simple equilibrium model. In a companion paper, Yves Balasko (forthcoming) investigates a more general formulation of the notion of extrinsic uncertainty. For an overview of the subject of rational expectations equilibria in which extrinsic uncertainty matters, see our propaganda piece, Cass and Shell (1980).

the natural restrictions which must be imposed on market participation—namely, that consumers born after sunspot activity is observed cannot possibly trade on the securities market meant to provide the possibility of hedging against the effects of this uncertainty. Section III also provides a complete description of consumer behavior in the securities and commodities markets.

Section IV defines and analyzes market equilibrium. This section might appear to be our mathematical ghetto. Indeed, there is a good deal of notation (which is the only way we know to make these ideas precise!), but the arguments themselves are not very complicated. Equilibrium in the economy with securities and commodities markets is shown to be equivalent to equilibrium in an economy with contingent-claims markets. (This equivalence is the one established by Arrow [1964].) The latter formulation of market equilibrium is the more tractable and is exclusively employed in the succeeding analysis. The traditional general equilibrium model yields "certainty equilibria" in which sunspots cannot matter. It is shown that, in principle, our market equilibrium concept is broader than that of the traditional model; certainty equilibrium is a special case of market equilibrium.

Section V is devoted to the degenerate—and contrafactual—case in which there are no restrictions on market participation. If consumers share the same probability beliefs, then sunspots do not matter; that is, sunspot activity does not affect the allocation of resources. In this case, market equilibrium is equivalent to certainty equilibrium. If, however, subjective probabilities differ across consumers, then sunspots are bound to matter. These ideas are presented in a series of three Edgeworth box diagrams in Section V.

The reader should seek the return on his investment in Section VI, where the consequences of restricted market participation are analyzed. Attention is focused on the more difficult (not necessarily the more realistic) case of shared probability beliefs. There is always a market equilibrium in which sunspots do not matter, that is, a market equilibrium equivalent to a certainty equilibrium. But we also demonstrate that in fact our market equilibrium concept is broader than the traditional certainty equilibrium concept. In Section VI, we give simple examples in which sunspot equilibria are lotteries over certainty equilibria. Randomization over certainty equilibria is not, however, the only possible source of sunspot equilibria. A more subtle example presented in the Appendix possesses a unique certainty equilibrium and (at least) one sunspot equilibrium.

Section VII is devoted to the welfare analysis. Equilibrium allocations in which sunspots do not matter are Pareto optimal in the traditional sense. While equilibrium allocations in which sunspots do matter are not Pareto optimal in this sense, they are Pareto optimal in a natural, weaker sense, where each consumer is identified by his state of birth (as well as by the other usual parameters—namely, tastes and endowments).

The upshot of this analysis is that in a rational expectations, general equilibrium world, the presence of extrinsic uncertainty—sunspots, waves of pessimism/optimism, and so forth—may well have real effects. The lesson for macroeconomics is that, even if one assumes the most favorable informational and institutional conditions imaginable, there may be a role for the government to stabilize fluctuations arising from seemingly noneconomic disturbances.

II. Extrinsic Uncertainty

The basic parameters defining an economy—such as endowments, preferences, and production possibilities—are referred to as the fundamentals of that economy. We make a sharp distinction between those things which influence the fundamentals and those which do not. If a variable has an effect on the fundamentals, we say that it is *intrinsic* (to the economy); otherwise, the variable is said to be *extrinsic*. Our basic question is: What effect can an extrinsic variable, say, sunspot activity, have on the allocation of resources?

Before going further, we must introduce some elements of our model; even the concept of extrinsic variable remains vague until it is applied in a formal framework. The model we adopt in this paper is the simplest possible for our purpose, but our results generalize quite readily. The more general arguments can be found in our working paper (Cass and Shell [1981], esp. app. 1, pp. 29–39).

We assume that there are two standard commodities, $i = 1, 2$, and two possible states of nature, $s = \alpha, \beta$. Let $x_h^i(s)$ denote consumption by consumer h of commodity i in state s, let $x_h(s)$ denote the vector of state-s consumptions, $(x_h^1(s), x_h^2(s))$, and let x_h denote the vector of prospective consumptions, $(x_h(\alpha), x_h(\beta)) = (x_h^1(\alpha), x_h^2(\alpha), x_h^1(\beta), x_h^2(\beta))$. Consumer h is endowed with prospective goods, denoted by the strictly positive vector $\omega_h = (\omega_h(\alpha), \omega_h(\beta)) = (\omega_h^1(\alpha), \omega_h^2(\alpha), \omega_h^1(\beta), \omega_h^2(\beta))$. The preferences of Mr. h are described by the utility function, $v_h(x_h)$, which is defined over his prospective consumption plans. There is no production in our simple economy; the fundamentals are endowments and preferences. We assume that uncertainty is purely extrinsic, allowing us to think of the random variable s as representing sunspot activity, identifying α with the state "sunspots" and β with the state "no sunspots."

Endowments are not affected by sunspot activity, that is,

$$\omega_h(\alpha) = \omega_h(\beta) \tag{1}$$

for each h. Conditions must also be imposed on preferences before we can say that uncertainty is extrinsic. We assume that consumer behavior is based on von Neumann–Morgenstern utilities. Consumer h believes that sunspots occur with probability $\pi_h(\alpha)$ and that no sunspots occur with probability $\pi_h(\beta) = 1 - \pi_h(\alpha)$. Preferences are then represented by the familiar expected utility criterion, that is,

$$v_h(x_h(\alpha), x_h(\beta)) = \pi_h(\alpha)u_h(x_h(\alpha)) + \pi_h(\beta)u_h(x_h(\beta)) \tag{2}$$

for each h. We assume that the functions u_h are smooth, strictly increasing, and strictly concave, implying that Mr. h exhibits strict risk aversion in his evaluation of lotteries. Underlying tastes—as described by the function u_h—are obviously independent of sunspot activity, since the only effect of s on u_h is through its effect on the allocation $x_h(s)$; because of (2), sunspot activity, unlike (say) rainfall, has no direct effect on the consumer's well-being. Since specifications (1) and (2) ensure that s is extrinsic, we are justified in referring to it as a measure of sunspot activity.

It is no surprise that underlying tastes are unaffected by uncertainty for von Neumann–Morgenstern consumers. In the axioms defining von Neumann–Morgenstern behavior, uncertainty is no more than the basis for constructing lotteries. We adopt the specification (2) partly because the expected utility hypothesis is so familiar, but we have another reason.

Equations (1) and (2) imply that uncertainty is extrinsic, but taken together these conditions are more severe than is necessary. In particular, von Neumann–Morgenstern preferences are not the only ones which are unaffected by uncertainty. The more basic assumption would be that the consumer is indifferent between otherwise identical lotteries generated by different random variables. The utility function defined over prospective consumption plans is parameterized by subjective probabilities. If Mr. h is concerned only with lotteries and is not affected by sunspots per se, his preferences satisfy

$$v_h(x_h(\alpha), x_h(\beta); \pi_h(\alpha), \pi_h(\beta)) = v_h(x_h(\beta), x_h(\alpha); \pi_h(\beta), \pi_h(\alpha)). \tag{3}$$

That is, Mr. h is interested only in payoffs and their respective probabilities—not in the "names" given to the states of nature.

Our assumption (2) clearly entails (3). And it entails more. The separability introduced in (2) implies that Mr. h's consumption plan will be intertemporally consistent, that with perfect foresight he will not revise his plan after uncertainty has been revealed. (See Donaldson and Selden [1981] for further elaboration of this point.) Intertemporal consistency is essential to much of our analysis.

In most of the sequel—but the exceptions are significant—we assume that beliefs about sunspot activity are the same for each consumer, that is,

$$\pi_h(\alpha) = \pi(\alpha) \qquad \text{so that } \pi_h(\beta) = \pi(\beta) \tag{4}$$

for each h. The commonly held probabilities, $\pi(\alpha)$ and $\pi(\beta)$, can be thought of as objective probabilities, but that is not necessary. Equations (4) can also be taken as a strong version of the rational expectations hypothesis. A good case can be made against the realism of this assumption. In part, however, imposing (4) is like tying one hand behind one's back. Without (4) the point we shall make is easy to establish,[3] but even with (4) we shall establish that sunspot activity can affect the allocation of resources.

We say that extrinsic uncertainty matters to the allocation of resources (or, simply, *sunspots matter*) if some consumer's allocation depends on the state of nature, that is,

$$x_h(\alpha) \neq x_h(\beta) \tag{5}$$

for some h. If, on the other hand, all consumers' allocations are independent of the state of nature, that is,

$$x_h(\alpha) = x_h(\beta) \tag{6}$$

for each h, then we say that extrinsic uncertainty does not matter to the allocation of resources (or simply, *sunspots do not matter*). These definitions will be restated in terms of prices, instead of the quantities x_h, after market behavior has been introduced. The definitions could also be restated in terms of utilities. For example, sunspots would not matter if $u_h(x_h(\alpha)) = u_h(x_h(\beta))$ for each h. The utility-based definitions would be more appropriate in a general setting, but it turns out that these three ways of defining whether or not sunspots matter are equivalent under our simplifying assumptions of strict risk aversion and smooth utility functions.[4]

III. Market Participation

It is often argued that the Arrow-Debreu general equilibrium model is remarkably versatile. In particular, it is claimed that this model can be employed to analyze intertemporal market trading in the face of uncertainty; one needs only to define commodities appropriately to

[3] See the discussion based on fig. 3 in Sec. V.

[4] The utility-based definition does not include as sunspot equilibria trivial cases involving moving along flats in indifference curves (which would be included under the corresponding quantity-based definition) or rotating around kinks in indifference curves (which would be included under the corresponding price-based definition). The utility-based definition is given in the context of a general model in Cass and Shell (1981, app. 1, esp. pp. 34–35).

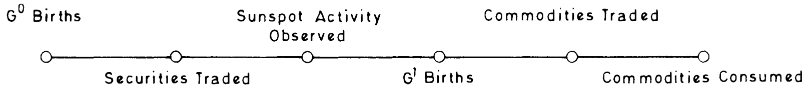


FIG. 1.—The time line

account for time and states of nature. There is, however, one fundamental and significant aspect of actual dynamic economies which is not reflected in the usual Arrow-Debreu framework. In the real world, the market trading process itself takes place in time. Trades can include promises to deliver commodities in the future under specific circumstances, but each of the parties to a trade must be alive on the trading date. Even if currently alive individuals can know today the prices which will prevail in the future, they simply cannot trade today with individuals whose birth dates are in the future.

Even in a world where birth dates and death dates vary across individuals, one can readily conceive of the existence of a complete array of markets. What cannot be imagined, however, is that there could be unrestricted participation in these markets. At any given time, some of the potential actors have already left the stage, while others have yet to come onstage. The distinction between the assumptions of complete markets and unrestricted market participation is important. Throughout the present paper, we assume that markets are complete. We show that the essential nature of the equilibrium allocation process is altered when the natural restrictions on market participation are incorporated into the analysis.

We now return to the formal model. Consumers were introduced in the previous section. Here, we place those consumers in time and then describe the market structure for the abstract economy. The timing of trades is of critical importance. We urge the reader to rely heavily on figure 1, our time line.

We assume that there are two generations. Consumers in generation 0 (G^0, for short) are born at the "beginning of time" and live to the "end of time." Consumers in generation 1 (G^1, for short) are born later than those in G^0 but, like those in G^0, live to the end of time. (This simple demography provides the basis for a finite-horizon overlapping-generations model.) The consumers in G^0 are born before sunspot activity is revealed. They can trade with each other on the market for securities which are contingent on the outcome of the extrinsic random variable, sunspot activity. They can also trade with each other and members of G^1 on the spot market for commodities, which meets after the observation of sunspot activity. On the other hand, consumers in G^1 are born after the degree of sunspot activity is known. They can trade with each other and consumers from G^0 on the spot commodities market, but, of course, they cannot trade on the securities market, which must convene before they are born. Thus, participation in the commodities market is unrestricted, while participation in the securities market is necessarily restricted.

Three assets can be traded on the securities market. Money, all of which is issued by consumers, is the unit of account. Without loss of generality, the nominal interest rate on money can be set equal to zero, so that a consumer who purchases a dollar on the securities market is able to exchange it on the spot market for one dollar's worth of commodities. A consumer who has issued a dollar of (inside) money on the securities market can pay off this debt by delivering a dollar's worth of commodities on the spot market. Let m_h be the purchases (or sales, if negative) of inside money by consumer h (in G^0) on the securities market.

A unit of money represents a promise which entitles the holder to a dollar's worth of commodities no matter what state of nature is observed. In our model, money is interesting only as a unit of account. There is no need for a pure store of value in this simple economy. Nor does money serve as a hedge against uncertainty. We next describe the mechanism by which consumers in G^0 can hedge against sunspot activity.

Consumers in G^0 are assumed to be able to buy and sell securities, the payoffs from which are contingent on sunspot activity. Consumer h is assumed to purchase $b_h(s)$ units of securities, each unit of which is worth one dollar on the spot market if state s has been observed but is worthless otherwise. The dollar price on the securities market of one unit of this security is $p_b(s)$, $s = \alpha, \beta$. Let b_h denote the vector of contingent securities purchased by consumer h, that is, $b_h = (b_h(\alpha), b_h(\beta))$, and let p_b denote the price vector $(p_b(\alpha), p_b(\beta))$.

Both generations trade on the spot market where the dollar price of commodity i, given that state s has been observed, is $p_c^i(s)$, $s = \alpha, \beta$, and $i = 1, 2$. Let the commodity price vector $(p_c^1(s), p_c^2(s))$ be denoted by $p_c(s)$ and let p_c denote the price vector $(p_c(\alpha), p_c(\beta)) = (p_c^1(\alpha), p_c^2(\alpha), p_c^1(\beta), p_c^2(\beta))$.

We have by now accumulated sufficient notation to be able to describe formally the opportunities and the behavior of consumers. Opportunities, and thus behavior, are substantially different for consumers in different generations.

Consumer h in G^0 chooses a plan (x_h, m_h, b_h) which maximizes his expected utility subject to his market constraints, that is, which solves

maximize

$$\pi(\alpha)u_h(x_h(\alpha)) + \pi(\beta)u_h(x_h(\beta))$$

subject to

$$m_h + p_b \cdot b_h = 0,$$
$$p_c(\alpha) \cdot x_h(\alpha) \leqq p_c(\alpha) \cdot \omega_h(\alpha) + m_h + b_h(\alpha), \tag{7'}$$
$$p_c(\beta) \cdot x_h(\beta) \leqq p_c(\beta) \cdot \omega_h(\beta) + m_h + b_h(\beta),$$

and

$$x_h \geqq 0$$

for h in G^0. The dot denotes inner product; thus, for example, $p_b \cdot b_h = p_b(\alpha)b_h(\alpha) + p_b(\beta)b_h(\beta)$ and $p_c(\alpha) \cdot x_h(\alpha) = p_c^1(\alpha)x_h^1(\alpha) + p_c^2(\alpha)x_h^2(\alpha)$.

Discussion of the constraints in (7′) is in order. The first constraint tells us that consumers in G^0 must finance their purchases of securities from their sales of other securities. This seems to us to be a realistic assumption, but it is not essential. In the next section, we show that consumers in G^0 behave as if they were facing a complete array of Arrow-Debreu contingent-claims markets. The first constraint is written as an equality, which suggests nondisposability of securities. This is a technical maneuver made to simplify the analysis. The second and third constraints in (7′) are based on an assumption of rational expectations: Sunspot activity is uncertain, but consumers in G^0 have perfect foresight about the prices which will prevail in each of the two states of nature. Then, in each state of nature, commodity purchases are financed by the sales of other commodities, net holdings of inside money, and net holdings of the security which pays off in that particular state.

Life is relatively simple for consumers in G^1. Before they are born, sunspot activity has been observed. These consumers do not have to make decisions under uncertainty, nor do they need to form expectations about prices. A consumer h in G^1 chooses a consumption vector $x_h(s)$ which maximizes his utility subject to his market constraints, that is, which solves

maximize

$$u_h(x_h(s))$$

subject to

$$p_c(s) \cdot x_h(s) \leqq p_c(s) \cdot \omega_h(s) \tag{8'}$$

and

$$x_h(s) \geqq 0$$

for $s = \alpha, \beta$ and h in G^1.

Whichever state of nature has occurred, the consumer in G^0 maximizes his utility. His consumption is financed from the market value of his endowments.

The market behavior of consumers is completely described by the system (7′) and (8′). The next section treats market equilibrium.

IV. Market Equilibrium

A market (or competitive) equilibrium can be defined by a system of prices for which consumer demands are consistent with constrained utility maximization and for which materials balance. Formally, then, a market equilibrium is a positive price vector $(p_b, p_c) = (p_b(\alpha), p_b(\beta), p_c(\alpha), p_c(\beta))$, which solves the system

$$\sum_{G^0} m_h = 0,$$

$$\sum_{G^0} b_h(s) = 0, \tag{9'}$$

and

$$\sum_{H} x_h(s) \leqq \sum_{H} \omega_h(s),$$

where $(m_h, b_h(s), x_h(s))$, $s = \alpha, \beta$, is the solution to (7′) for h in G^0, $x_h(s)$ is the solution to (8′) for h in G^1, and H is the set of all consumers, that is, the union of G^0 and G^1, which implies, for example, that

$$\sum_{H} x_h(s) = \sum_{G^0} x_h(s) + \sum_{G^1} x_h(s).$$

The equations in (9′) require that for each of the three (nondisposable) securities (including money) demand is equated to supply. The sums in these equations are taken over G^0, reflecting two of our assumptions: (i) participation in these markets is not possible for consumers in G^1; and (ii) there is neither outside (i.e., government) money nor outside securities. Money taxes and transfers could have been included. However, since the time horizon is finite, unless the sum of taxes exactly equals the sum of transfers, the equilibrium price of money would necessarily be zero. This is the well-known problem of "hot-potato" money, which can be avoided in models with an unbounded time horizon (see, e.g., Cass and Shell 1980).

The inequality in (9′) states that aggregate demand for each commodity in each state cannot exceed the corresponding aggregate supply. The sums are taken over H, since there are no restrictions on commodity-market participation.

An immediate implication of equilibrium in the securities market can be drawn. Each consumer in G^0 must be indifferent between holding one unit of money or one unit of each of the two contingent securities. Hence, in equilibrium, the price of the composite commodity "one unit of sunspot security *and* one unit of nonsunspot security" must be equal to the price of money. This idea is formalized in the following proposition.

PROPOSITION 1: A market equilibrium price vector $(p_b(\alpha), p_b(\beta), p_c(\alpha), p_c(\beta))$ must satisfy $p_b(\alpha) + p_b(\beta) = 1$.

PROOF: Assume that $p_b(\alpha) + p_b(\beta) < 1$. Each consumer in G^0 is then able to sell money and purchase more units of each of the contingent securities than the number of dollars which he sold. Consider, for instance, financial plans $(m_h, b_h(\alpha), b_h(\beta))$ with the property that $b_h(s) = (-m_h)/(p_b(\alpha) + p_b(\beta))$ for $s = \alpha, \beta$. Such plans clearly satisfy the equality constraint in (7′). Furthermore, for each number N, there is a plan from this class which satisfies $m_h + b_h(s) > N$ for $s = \alpha, \beta$. Hence, the right-hand sides of the second and third constraints in (7′) can be made arbitrarily large, so that consumers in G^0 can enjoy arbitrarily large consumption in each of the two states. This contradicts the inequality in (9′); consumers in H can supply at most a finite amount of the commodities. Assuming $p_b(\alpha) + p_b(\beta) > 1$ leads to a similar contradiction. In this case, consumers in G^0 are able to make arbitrarily large the right-hand sides of the second and third constraints in (7′) by selling each of the two contingent securities in equal (but large) quantities while using the revenue to purchase money. The proof by contradiction is complete.

Proposition 1 is a consequence of the fact that, in equilibrium, there can be no pure arbitrage-profit opportunities on the securities market. The proof of proposition 1 also establishes that, at equilibrium prices, a consumer in G^0 can choose to buy or sell any quantity of money without constraining his consumption opportunities by this choice. Suppose that at equilibrium prices $(p_b(\alpha), p_b(\beta), p_c(\alpha), p_c(\beta))$, the vector $(x_h^*(\alpha), x_h^*(\beta), m_h^*, b_h^*(\alpha), b_h^*(\beta))$ is a particular solution to the constrained utility maximization problem (7′). Consider another value of money purchases, m_h^{**}. Since $p_b(\alpha) + p_b(\beta) = 1$, it follows from (7′) that $(x_h^*(\alpha), x_h^*(\beta), m_h^{**}, b_h^{**}(\alpha), b_h^{**}(\beta))$ is also utility maximizing if $b_h^{**}(s) + m_h^{**} = b_h^*(s) + m_h^*$, $s = \alpha, \beta$. This allows us to think of the contingent securities as state-specific moneys.

An important assumption is implicit in the choice of (ordinary) money as the unit of account (cf. [7′]). In any properly specified monetary economy, there is always an equilibrium in which the commodity price of money is zero (see Cass and Shell 1980; Balasko and

Shell 1981). In order to simplify matters here, we have excluded this equilibrium. If the price of money is zero, then the prices of contingent securities must also be zero. This amounts to assuming that the securities market is closed, so that, in effect, all consumers are restricted from the securities market, which yields essentially the same equilibria as in the model where $H = G^1$.

Our contingent securities are exactly those studied by Arrow (1964). When perfect foresight and von Neumann–Morgenstern utilities are assumed, the traditional model with complete (Arrow) securities markets and complete spot markets is equivalent to the traditional model with complete (Arrow-Debreu) contingent-claims markets. This insight allows us to replace the equilibrium system (7′)–(9′) with another that turns out to be easier to analyze. The analysis will be in terms of the new, more compact model, but the realism of our formulation is best judged by the extended model, (7′)–(9′). We next formulate the reduced-form model and establish its equivalence to the extended model.

The price of a contract to deliver one unit of commodity $i = 1, 2$ if state of nature $s = \alpha, \beta$ occurs is denoted by $p^i(s)$. We denote the vector of state-s prices $(p^1(s), p^2(s))$ by $p(s)$ and the full vector of contingent-claims prices $(p(\alpha), p(\beta)) = (p^1(\alpha), p^2(\alpha), p^1(\beta), p^2(\beta))$ by p. Consumer h in G^0 chooses a prospective consumption plan $x_h = (x_h(\alpha), x_h(\beta))$ that maximizes his expected utility subject to his market constraints, that is, that solves

maximize

$$\pi(\alpha)u_h(x_h(\alpha)) + \pi(\beta)u_h(x_h(\beta))$$

subject to

$$p \cdot x_h \leqq p \cdot \omega_h \tag{7}$$

and

$$x_h \geqq 0$$

for h in G^0. The value of his prospective consumption plan $p \cdot x_h = p(\alpha) \cdot x_h(\alpha) + p(\beta) \cdot x_h(\beta)$ must not exceed the value of his prospective endowments, $p \cdot \omega_h = p(\alpha) \cdot \omega_h(\alpha) + p(\beta) \cdot \omega_h(\beta)$. Consumers from G^0 are also able to trade on the market for spot commodities at prices $p_c(s)$ for $s = \alpha, \beta$. In the full-blown version of this model,[5] an equilibrium condition (in fact, a zero-arbitrage-profit condition analogous to that described in proposition 1) is that the vectors $p(s)$ and $p_c(s)$ be proportional, so that without loss of generality we can set $p_c(s) = p(s)$

[5] See Cass and Shell (1981, sec. III). The conditions of intermarket equilibrium are established in lemma 1, pp. 16–17.

for $s = \alpha, \beta$. Thus, the consumption plan x_h which solves (7) is the same as the one in which there is also a spot market but where prices are such that no profits can be made by arbitraging between the two markets. Setting the spot price $p_c^i(s)$ equal to the contingent-claims price $p^i(s)$, $i = 1, 2$ and $s = \alpha, \beta$, allows us to describe the behavior of consumers in G^1 in a fashion consistent with (7).

Consumer h in G^1 trades only on the spot market. He chooses for each state of nature—after it has been revealed—a consumption bundle $x_h(s)$ that maximizes his utility subject to his market constraints, that is, that solves

maximize

$$u_h(x_h(s))$$

subject to

$$p(s) \cdot x_h(s) \leqq p(s) \cdot \omega_h(s) \tag{8}$$

and

$$x_h(s) \geqq 0$$

for $s = \alpha, \beta$ and h in G^1.

A market equilibrium for the reduced (contingent-claims) economy is a positive price vector $p = (p(\alpha), p(\beta))$ that solves the system

$$\sum_H x_h \leqq \sum_H \omega_h, \tag{9}$$

where $x_h = (x_h(\alpha), x_h(\beta))$ is the solution to (7) for h in G^0 and to (8) for h in G^1. Inequality (9) is the materials balance condition.

The prices in the system (7′)–(9′) are dollar prices. There is no obvious unit of account for the reduced system (7)–(9). We have not chosen a normalization. If p solves (7)–(9), then so does any nonnegative vector proportional to p. Thus, if p is unique up to a scalar multiple, then we say that the equilibrium is unique.

We establish in the following proposition that the economy described by (7)–(9) is in all essentials equivalent to the one described by (7′)–(9′) or, more accurately, that the former is a reduced-form version of the latter.

PROPOSITION 2: (i) If the price vector $(p_b, p_c) = (p_b(\alpha), p_b(\beta), p_c(\alpha), p_c(\beta))$ is a market equilibrium for the extended economy described by (7′)–(9′), then any nonnegative price vector $p = (p(\alpha), p(\beta))$ which is proportional to the vector $(p_b(\alpha)p_c(\alpha), p_b(\beta)p_c(\beta))$ is a market equilibrium for the reduced economy described by (7)–(9). (ii) If $p = (p(\alpha), p(\beta))$ is a market equilibrium for the reduced economy described by

(7)–(9), then there is a nonnegative price vector $(p_b, p_c) = (p_b(\alpha), p_b(\beta), p_c(\alpha), p_c(\beta))$, with $(p_c(\alpha), p_c(\beta))$ proportional to $(p(\alpha)/p_b(\alpha), p(\beta)/p_b(\beta))$, which is a market equilibrium for the economy described by (7′)–(9′).

PROOF: (i) Assume that (p_b, p_c) is a solution to (9′) and that the corresponding equilibrium allocation for consumer h in H is x_h. Set p equal to $\lambda[p_b(\alpha)p_c(\alpha), p_b(\beta)p_c(\beta)]$, where λ is a positive scalar. Since $\lambda p_b(s)$ is a positive scalar for $s = \alpha, \beta$, p and x_h for h in G^1 satisfy (8). The clearing condition (9′) entails (9), so it remains to show that p and x_h for h in G^0 satisfy (7). The first three constraints in (7′) reduce to

$$(p_b(\alpha)p_c(\alpha)) \cdot (x_h(\alpha) - \omega_h(\alpha)) - p_b(\alpha)m_h \leqq p_b(\alpha)b_h(\alpha)$$

and

$$(p_b(\beta)p_c(\beta)) \cdot (x_h(\beta) - \omega_h(\beta)) + (1 - p_b(\beta))m_h \leqq -p_b(\alpha)b_h(\alpha).$$

We can take $b_h(\alpha)$ to be an unconstrained slack variable for the system above. Therefore, after multiplying by λ and using proposition 1, we can combine the two inequalities above to yield a more compact description of the constraints on the consumption plan of h in G^0, namely, $p \cdot x_h \leqq p \cdot \omega_h$. The proof of part i is complete. (ii) Assume that p is a solution to (9) and that the corresponding equilibrium allocation for consumer h in H is x_h. Set p_c equal to $\lambda[p(\alpha)/p_b(\alpha), p(\beta)/p_b(\beta)]$, where λ is a positive scalar, and $p_b(\alpha)$ and $p_b(\beta)$ are positive fractions satisfying $p_b(\alpha) + p_b(\beta) = 1$. Then for h in G^1, x_h obviously satisfies (8′) at spot prices p_c. Next turn to (7′). Set $m_h = 0$ for h in G^0—which can be done without sacrificing any consumption opportunities for Mr. h—so that by construction the money market equilibrium condition in (9′), $\Sigma_{G^0} m_h = 0$, is satisfied. Remember that the allocation x_h solves (7) at prices p. Solve the equation

$$\lambda p(s) \cdot (x_h(s) - \omega_h(s)) = p_b(s)b_h(s) \tag{10}$$

for the sole unknown, $b_h(s)$, $s = \alpha, \beta$. From (8), (9), and nonsatiation, we have that

$$\sum_{G^0} p(s) \cdot (x_h(s) - \omega_h(s)) = 0$$

and therefore, from (10), that the securities market equilibrium condition in (9′), $\Sigma_{G^0} b_h(s) = 0$, is satisfied. It follows from (7) and nonsatiation that, for h in G^0,

$$p(\alpha) \cdot (x_h(\alpha) - \omega_h(\alpha)) = -p(\beta) \cdot (x_h(\beta) - \omega_h(\beta)),$$

which, combined with (10), yields $p_b(\alpha)b_h(\alpha) + p_b(\beta)b_h(\beta) = 0$, so that the first constraint in (7′) is satisfied. Furthermore, that constraint and (10) yield the system of equations

$$\begin{aligned} \lambda p(\alpha) \cdot (x_h(\alpha) - \omega_h(\alpha)) &= p_b(\alpha) b_h(\alpha) \\ \lambda p(\beta) \cdot (x_h(\beta) - \omega_h(\beta)) &= -p_b(\alpha) b_h(\alpha), \end{aligned} \tag{11}$$

where $b_h(\alpha)$ is a slack variable, which yields the result that, in the commodity space, the budget set implied by (11) is the same as that implied by the (equality form of the) single budget constraint in (7). Hence, if x_h is a solution to (7) at prices p, then $(x_h, m_h = 0, b_h)$ is a solution to (7′) at prices (p_b, p_c). Finally, it follows from (9) that the materials balance condition, $\Sigma_H x_h \leqq \Sigma_H \omega_h$, is also satisfied. The proof of part ii is complete.

In the sequel, the (simpler) reduced form of the model, (7)–(9), replaces the (seemingly more realistic) extended form of the model (7′)–(9′).

In Section II, we defined whether or not sunspots matter in terms of commodity allocations. An alternative definition, based on the equilibrium prices $p = (p(\alpha), p(\beta))$, can now be given. Because of the invariant endowment assumption (1), the von Neumann–Morgenstern utility hypothesis (2), the rational expectations postulate (4), and the strict risk-aversion assumption, we can say that sunspots matter if $(p(\alpha)/\pi(\alpha)) \neq (p(\beta)/\pi(\beta))$; otherwise, sunspots do not matter.

Prior to our fascination with sunspots (and other instances of extrinsic uncertainty), most general equilibrium economists and many commonsense economists had an instinct about what role should be assigned to sunspots in economic models: Ignore them! We refer to the parallel system in which sunspots are ignored as the certainty model; the corresponding equilibrium is, then, the certainty equilibrium. We next formally define certainty equilibrium, allowing a comparison to be made between this equilibrium concept and ours.

Consider the economy described in the preceding pages, but ignore (extrinsic) uncertainty. There are only two commodities. Mr. h in H consumes y_h^1 units of the first and y_h^2 units of the second. His corresponding endowments of the commodities are w_h^1 and w_h^2. The prices of the commodities are q^1 and q^2. Define the vectors y_h, w_h, and q by $y_h = (y_h^1, y_h^2)$, $w_h = (w_h^1, w_h^2) = \omega_h(s)$ for $s = \alpha, \beta$ from (1), and $q = (q^1, q^2)$. Mr. h chooses y_h in order to maximize his utility subject to his budget constraint, that is, to solve

maximize

$$u_h(y_h)$$

subject to

$$q \cdot y_h \leqq q \cdot w_h \tag{12}$$

and

$$y_h \geqq 0$$

for h in H. A *certainty equilibrium* is a positive price vector q satisfying materials balance, that is,

$$\sum_H y_h \leqq \sum_H w_h, \tag{13}$$

where y_h is the solution to (12) for h in H.

Notice that each certainty equilibrium q is equivalent to the market equilibrium $p = (p(\alpha), p(\beta))$ defined by $p(s) = \pi(s)q$, $s = \alpha, \beta$. Furthermore, the corresponding allocations, y_h and $x_h = (x_h(\alpha), x_h(\beta))$, satisfy $y_h = x_h(s)$ for $s = \alpha, \beta$. Thus, every certainty equilibrium q corresponds to some market equilibrium p. Are there other equilibria which satisfy (7)–(9)? This question is answered in the next two sections.

V. Unrestricted Market Participation

A basic theme of this paper is that in a truly dynamic economy, in which generations overlap and in which there is (at least) potential uncertainty, market participation is necessarily restricted. Viewed from this perspective, the present section, in which we analyze an economy with unrestricted market participation, would seem to be a detour. This is not the case. Even with complete participation, our equilibrium concept (described formally by [7]–[9]) is potentially different from the classical (certainty) equilibrium concept (described formally by [12]–[13]). We establish in this section that sunspots do not matter in economies with strong rational expectations, complete markets, and unrestricted access to those markets. Under these strong assumptions, equilibrium in our model is thus essentially equivalent to equilibrium in the classical model. The strong assumptions are important. In particular, if consumers do not share the same beliefs about sunspot activity, then sunspots are bound to matter—even with perfect markets.

The assumption of unrestricted participation means that there are no consumers in G^1; G^1 is empty. Then, the set of all consumers H is identical with the set of unrestricted consumers G^0. Our formal model is then completely described by (7) and (9). We next state our fundamental result on the structure of equilibrium with unrestricted market participation; assumption (4) is critical.

PROPOSITION 3: If market participation is unrestricted, then there is no equilibrium $p = (p(\alpha), p(\beta))$ in which sunspots matter.

PROOF: Suppose otherwise. That is, assume that there is an equilibrium in which $(p(\alpha)/\pi(\alpha)) \neq (p(\beta)/\pi(\beta))$, which implies that $x_h(\alpha) \neq x_h(\beta)$ for h in $G^0 = H$. Consider the alternative allocation $\bar{x}_h = (\bar{x}_h(\alpha), \bar{x}_h(\beta))$ defined by "averaging over states,"

$$\bar{x}_h(s) = \pi(\alpha)x_h(\alpha) + \pi(\beta)x_h(\beta),$$

for $s = \alpha, \beta, h$ in H. The allocation $\bar{x}_h$ is independent of sunspot activity and is feasible (i.e., [9] is satisfied), since

$$\begin{aligned}\sum_H \bar{x}_h(s) &= \sum_H (\pi(\alpha)x_h(\alpha) + \pi(\beta)x_h(\beta)) \\ &= \pi(\alpha)\sum_H x_h(\alpha) + \pi(\beta)\sum_H x_h(\beta) \\ &\leqq \pi(\alpha)\sum_H \omega_h(\alpha) + \pi(\beta)\sum_H \omega_h(\beta) \\ &= \sum_H \omega_h(s),\end{aligned}$$

for $s = \alpha, \beta$, by (9) and (1). Because of the strict concavity of u_h, the alternative allocation $\bar{x}_h$ dominates the hypothesized equilibrium allocation x_h, that is,

$$\begin{aligned}v_h(\bar{x}_h(\alpha), \bar{x}_h(\beta)) &= \pi(\alpha)u_h(\bar{x}_h(\alpha)) + \pi(\beta)u_h(\bar{x}_h(\beta)) \\ &= \pi(\alpha)u_h(\bar{x}_h(\alpha)) + (1 - \pi(\alpha))u_h(\bar{x}_h(\alpha)) \\ &= u_h(\bar{x}_h(\alpha)) \\ &= u_h(\pi(\alpha)x_h(\alpha) + \pi(\beta)x_h(\beta)) \\ &> \pi(\alpha)u_h(x_h(\alpha)) + \pi(\beta)u_h(x_h(\beta)) \\ &= v_h(x_h(\alpha), x_h(\beta))\end{aligned}$$

for h in H. The inequality is strict since $x_h(\alpha) \neq x_h(\beta)$. We have thus established that there is a feasible allocation which (strictly) Pareto dominates the proposed equilibrium allocation. This is a contradiction of the well-known result that an equilibrium allocation in an Arrow-Debreu economy (such as the one described in [7] and [9]) is Pareto optimal.

A close reading of proposition 3 and its proof reveals that it can be taken as a direct adaptation for our purposes of an important theorem of Malinvaud (1972). Although condition (1) is extremely natural, it is stronger than is required for proposition 3. As in Malinvaud's analysis, the assumption that $\omega_h(\alpha) = \omega_h(\beta)$ for each h could be weakened to $\Sigma_H\omega_h(\alpha) = \Sigma_H\omega_h(\beta)$. See also Balasko (forthcoming).

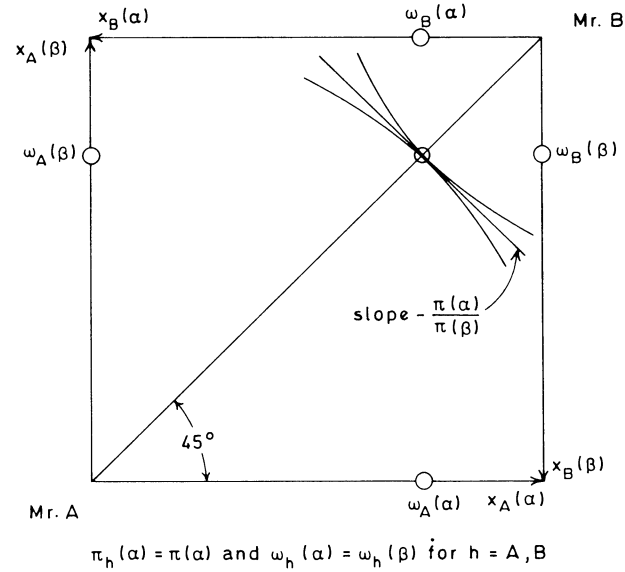
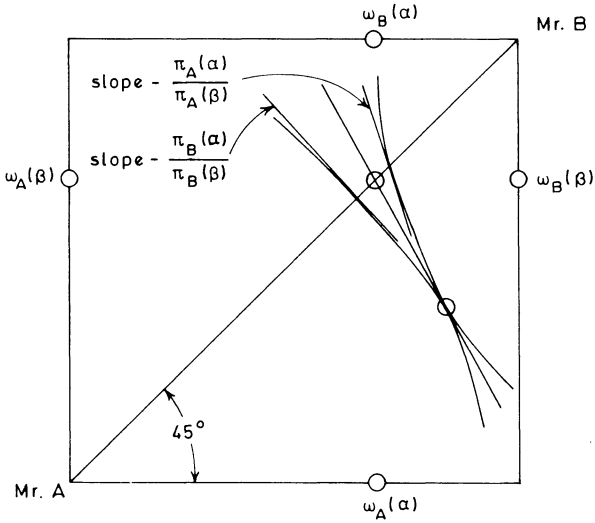


FIG. 2

Proposition 3 and related phenomena in the model with unrestricted participation are illustrated in figures 2–4. In drawing these box diagrams, we assume that there is only one commodity (rather than two) in each of the two states of nature. There are no consumers in G^1, and only two, Mr. A and Mr. B, in G^0.

Figure 2 is a diagrammatic illustration of proposition 3. This example underscores the fact that proposition 3 is based on the simple idea that rational, risk-averse individuals sharing the same beliefs will not take opposite sides of a bet, even if it is a fair bet. Allocations on the diagonal of the square yield tangency of indifference curves. Under the assumption of strict concavity, therefore, any allocation off the diagonal is Pareto dominated by some allocation on the diagonal. Consequently, since endowments lie on the diagonal, there are no mutually acceptable trades: Sunspots cannot matter.

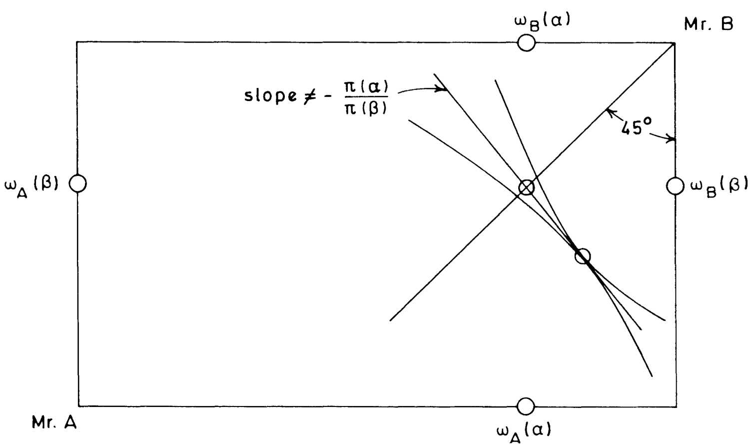
Proposition 3 is critically dependent on the assumption that consumers share common beliefs about the probability of sunspots. Differing beliefs are a powerful motivation for trading in contingent claims. In figure 3, once again endowments lie on the diagonal of the square. On the diagonal, the slope of Mr. h's indifference curve is given by the ratio of his subjective probabilities, $(\pi_h(\alpha)/\pi_h(\beta))$. Hence, if Mr. A and Mr. B assess different likelihoods, that is, $(\pi_A(\alpha)/\pi_A(\beta)) \neq (\pi_B(\alpha)/\pi_B(\beta))$, then each allocation on the diagonal is Pareto dominated. All equilibrium allocations must be off diagonal: When beliefs differ across individuals, extrinsic uncertainty is bound to matter. In



$\pi_A(\alpha) \neq \pi_B(\alpha),\ \omega_h(\alpha) = \omega_h(\beta)$ for $h = A, B$

FIG. 3

what follows, attention is restricted to the more intriguing case where beliefs are held in common—a very strong version of rational expectations. One should not, however, dismiss the practical importance of the consequences of the fact that rational individuals may have different forecasts because of different information. One of these consequences is that extrinsic uncertainty is likely to matter.



$\omega_A(\alpha) + \omega_B(\alpha) \neq \omega_A(\beta) + \omega_B(\beta)$

FIG. 4

Of course, uncertainty also matters if it affects preferences or aggregate endowments. In figure 4, Mr. A's endowments vary ceteris paribus with the state of nature. Since aggregate endowments then also vary with the state of nature, the box in figure 4 is not a square. Individual welfare can be improved if Mr. A shares some risk with Mr. B. Equilibrium is not at the autarky point.

Proposition 3 can also be extended to apply to uncertainty which does not affect tastes or aggregate endowments but does affect individual endowments. The reader can supply the argument by using figure 2. Mark an endowment which is off the diagonal. Then show that competitive equilibrium allocations must lie on the diagonal.

VI. Restricted Market Participation

In this section, we show that—even when probability beliefs are held in common—the traditional (certainty) equilibrium concept is too narrow. We first establish an equivalence between traditional equilibria and equilibria in which sunspots do not matter. We then show that there can be other market equilibria—equilibria in which sunspots matter.

The following proposition assumes shared probability beliefs but is valid for either restricted or unrestricted market participation.

PROPOSITION 4: There is always some market equilibrium in which sunspots do not matter. Indeed, market equilibria in which sunspots do not matter are equivalent to the traditional (certainty) equilibria.

PROOF: We assume that H is nonempty, but either G^0 or G^1 could be empty. (i) Let q be a solution to (13) with corresponding commodity allocations y_h. Then $p = (\pi(\alpha)q, \pi(\beta)q)$ is a solution to (9), since, given p, $x_h = (y_h, y_h)$ is a solution to (7) for h in G^0 and to (8) for h in G^1, respectively, if, given q, y_h is a solution to (12) for h in H. (ii) Let p be a solution to (9) such that sunspots do not matter, that is, such that $p(\alpha)/\pi(\alpha) = p(\beta)/\pi(\beta)$, with corresponding commodity allocations x_h, that is, so that $x_h(\alpha) = x_h(\beta)$. Then $q = p(\alpha)/\pi(\alpha)$ is a solution to (13), since, given q, $y_h = x_h(\alpha)$ is a solution to (12) for h in H if, given p, x_h is a solution to (7) (and therefore also [8]) for h in G^0 and to (8) for h in G^1, respectively. (iii) The existence of a market equilibrium in which sunspots do not matter follows from i together with the existence of a certainty equilibrium, which can be established by the usual fixed-point argument (see, e.g., Debreu 1959, chap. 5).

Thus, equilibrium allocations in which sunspots do not matter are unaltered as restrictions on securities market participation are introduced. What is altered is the set of potential market equilibrium allocations. The following observations demonstrate conclusively that, with restricted market participation, sunspots can matter.

OBSERVATION 1: Consider the case in which all consumers are excluded from participation in the securities market. Then there is an equilibrium in which sunspots matter if and only if there are at least two equilibria in which sunspots do not matter.

PROOF: G^0 is empty and thus $H = G^1$. (i) Assume that there are (at least) two distinct equilibria in which sunspots do not matter, say, p_j for $j = 1, 2$. Then $q_j = p_j(\alpha)/\pi(\alpha) = p_j(\beta)/\pi(\beta)$ is a certainty equilibrium for $j = 1, 2$. Moreover, $p = (q_1, q_2)$ is a market equilibrium, since it is a solution to (9) when $H = G^1$. Furthermore, because q_1 and q_2 represent distinct equilibria, $p(\alpha)/\pi(\alpha) \neq p(\beta)/\pi(\beta)$; that is, p is a market equilibrium in which sunspots matter. (ii) Suppose p is a market equilibrium and that $p(\alpha)/\pi(\beta) \neq p(\beta)/\pi(\beta)$. Then, the two price vectors $p_1 = (\pi(\alpha)p(\alpha), \pi(\beta)p(\alpha))$ and $p_2 = (\pi(\alpha)p(\beta), \pi(\beta)p(\beta))$ represent distinct equilibria in which sunspots do not matter because p_j is a solution to (9) when $H = G^1$ for $j = 1, 2$ while $\pi(\alpha)p(s)/\pi(\alpha) = \pi(\beta)p(s)/\pi(\beta)$ for $s = \alpha, \beta$.

In observation 1, we give a method for constructing sunspot equilibria. Consider any economy in which all market participation is restricted and in which there are multiple certainty equilibria and thus multiple nonsunspot equilibria. A sunspot equilibrium is constructed as a lottery over certainty equilibria. For example, consider such an economy with two distinct certainty equilibria, q_1 and q_2. No one is able to trade in securities, but sunspot activity has been recorded (in, say, its effects on rocks). Consumers believe that if there had been sunspots, the price vector $p(\alpha) = q_1$ would prevail; otherwise $p(\beta) = q_2$ would prevail. Consumers' price-expectation formation is rational. These prices prevail simply because people believe they will. Sunspots matter.

Observation 1 is based on a completely degenerate case (included mostly for purposes of motivation): All consumers are excluded from the securities market. The next observation is based on an economy in which there is a single consumer able to participate in the securities market and another consumer who is unable to participate in the securities market. (In fact, the observation remains valid when we assume that there are many consumers in G^1.) As in the previous example, sunspots can matter and sunspot equilibria are (only) based on lotteries over certainty equilibria.

OBSERVATION 2: In the two-consumer example described above, there is an equilibrium in which sunspots matter if and only if there are at least two equilibria in which sunspots do not matter.

PROOF: Let Mr. 0 be the consumer in G^0, Mr. 1 the consumer in G^1. (i) Assume there are (at least) two distinct nonsunspot equilibria, so there are two distinct certainty equilibria, q_1 and q_2. Let p be constructed as follows:

$$p = (\pi(\alpha)\lambda_1 q_1, \pi(\beta)\lambda_2 q_2), \tag{14}$$

where λ_j is the (optimal value of the Kuhn-Tucker-Lagrange) multiplier associated with the budget constraint in (12) for $h = 0$ when the price vector q is set equal to q_j for $j = 1, 2$. Then, on the one hand, the concave programming problem (12) for $h = 0$ has the associated Lagrangian

$$L_j(y_0, \lambda_j) = u_0(y_0) + (\lambda_j q_j) \cdot (w_0 - y_0)$$

for $j = 1, 2$, where $w_0 = \omega_0(s)$ for $s = \alpha, \beta$. On the other hand, when p satisfies (14), the concave programming problem (7) for $h = 0$ has the associated Lagrangian

$$\begin{aligned} L(x_0(\alpha), x_0(\beta), \mu) &= \pi(\alpha)u_0(x_0(\alpha)) + \pi(\beta)u_0(x_0(\beta)) \\ &\quad + \mu[\pi(\alpha)\lambda_1 q_1 \cdot (\omega_0(\alpha) - x_0(\alpha)) \\ &\quad + \pi(\beta)\lambda_2 q_2 \cdot (\omega_0(\beta) - x_0(\beta))] \\ &= \pi(\alpha)[u_0(x_0(\alpha)) + \mu\lambda_1 q_1 \cdot (\omega_0(\alpha) - x_0(\alpha))] \\ &\quad + \pi(\beta)[u_0(x_0(\beta)) + \mu\lambda_2 q_2 \cdot (\omega_0(\beta) - x_0(\beta))], \end{aligned}$$

where $\omega_0(s) = w_0$ for $s = \alpha, \beta$. We now appeal to the Kuhn-Tucker theorem for concave programming (see, e.g., Uzawa 1958). Given $q = q_j$, let y_{0j} be the optimal solution to (12) for $h = 0$, so that (y_{0j}, λ_j) is a saddle point to the Lagrangian function L_j for $j = 1, 2$. A simple computation (based on the fact that L is essentially the average of L_1 and L_2) verifies that at $(x_0(\alpha), x_0(\beta), \mu) = (y_{01}, y_{02}, 1)$ there is a saddle point to the Lagrangian function L. Hence, $(x_0(\alpha), x_0(\beta)) = (y_{01}, y_{02})$ must be the optimal solution to (7) for $h = 0$. Finally, it is trivial to show that $(x_1(\alpha), x_1(\beta)) = (y_{11}, y_{12})$ is the optimal solution to (8) for $h = 1$, so that p is a solution to (9). We have constructed an equilibrium in which sunspots matter. (ii) The proof of the necessity part is essentially equivalent to that provided in step ii of the proof of observation 1.

Observations 1 and 2 relate the existence of sunspot equilibria to the number of certainty equilibria (and, thus, to the number of nonsunspot equilibria). They are very particular to the specific assumptions employed (that there are no consumers in G^0 or only one consumer in G^0); they do not generalize in a meaningful way. The role of the two observations is simply to establish the existence of economic examples with perfect foresight and shared beliefs in which sunspots matter.

It should be remarked that, in rational expectations models of economies with outside money, we typically face a vast multiplicity of certainty equilibria (cf., e.g., Cass and Shell 1980; Balasko and Shell 1981). Whether or not this vast multiplicity has implications for the existence of monetary equilibria in which sunspots matter has yet to be investigated in a systematic way. While the results reported in observations 1 and 2 might suggest that sunspot equilibria are likely to be important in monetary economies, there are good reasons to be cautious in making any such conjecture.

Indeed, focusing on observations 1 and 2 could be misleading. In the specific models treated in this section, sunspot equilibria are in essence found by randomizing over (two) certainty equilibria. An example in the Appendix shows that this is by no means the only source of sunspot equilibria. For the model with two consumers in G^0 and one consumer in G^1, we construct endowments, beliefs, and preferences for which there is a unique certainty equilibrium but there is also (at least) one sunspot equilibrium. Obviously, then, the sunspot equilibrium cannot be thought of as a lottery over certainty equilibria. Moreover, the example clearly establishes that multiplicity of certainty (or nonsunspot) equilibria is not required for the existence of a sunspot equilibrium.

We close this section with our principal result on the possibility of sunspot equilibria.

PROPOSITION 5: In the rational expectations economy with natural restrictions on market participation, there can be equilibria in which sunspots matter.

PROOF: It follows from observation 1 or observation 2 and the well-known possibility that traditional (certainty) equilibria can be nonunique. The proposition also follows directly from the Appendix.

VII. Sunspots and Welfare

Normative analysis in the dynamic setting is not merely a simple extension of normative analysis based on the atemporal economy. For example, in infinite horizon models with overlapping finite-lived consumers but no uncertainty, the two fundamental theorems of welfare economics must be cast, not in terms of efficient or Pareto-optimal allocations, but rather in terms of short-run (or weakly) efficient or Pareto-optimal allocations.[6] Our present concern is not with the ef-

[6] See Cass (1972, esp. sec. III) and Balasko and Shell (1980, esp. sec. 4).

fects of an infinite horizon but, instead, with the consequences of combining generational overlap with riskiness. The crucial element which distinguishes our welfare problem is that, for a given risk, some people must literally live with it, while for others—those born after the uncertainty is resolved—it is only a historical datum, albeit a possibly important one.

In this section, two welfare standards are defined: the traditional Pareto criterion and a (weaker) dynamic Pareto criterion. Connections between the two criteria and market equilibrium allocation are explored. One reason for introducing the dynamic Pareto optimality concept is that it is the correct concept for proving welfare theorems about sunspot equilibria. We also believe there is more to it than that.

Consider Mr. h from G^1. At the beginning of time, the economic planner must consider the two potential selves of Mr. h, Mr. h-if-sunspots, denoted by $h(\alpha)$, and Mr. h-if-no-sunspots, denoted by $h(\beta)$. Suppose α occurs and Mr. h is given a meager allocation. He objects on the grounds of inequity. He is told that had β occurred, he would have received an ample allocation. The planner assessed his utility in ex ante terms. In those terms, he was well treated.[7] Will Mr. h, now Mr. $h(\alpha)$, accept this argument? Probably not. The planner might imagine that he could have been Mr. $h(\beta)$—or the Prince of Wales, for that matter—but, as far as he is concerned, he came into the world as Mr. $h(\alpha)$, and nothing can alter that fact. Furthermore, he points out (to gain allies and to heat up the debate) that, given the planner's ex ante point of view, there would be no pure equity argument in favor of the Equal Rights Amendment. Before conception, were not each of us potentially male and potentially female?

Mr. $h(\alpha)$'s (ex post) point of view, that he is separate from Mr. $h(\beta)$, motivates our definition of dynamic Pareto optimality. The (ex ante) point of view of our (straw man) planner motivates our definition of Pareto optimality (in the traditional sense).

An allocation $x = (\ldots, x_h, \ldots)$, where $x_h = (x_h(\alpha), x_h(\beta))$, is said to be *dynamically Pareto optimal* if there is no other allocation $x' = (\ldots, x'_h, \ldots)$ with the properties:

$$v_h(x'_h) \geqq v_h(x_h) \qquad \text{for } h \text{ in } G^0$$

and (i)

$$u_h(x'_h(s)) \geqq u_h(x_h(s)) \qquad \text{for } h \text{ in } G^1 \text{ and } s = \alpha, \beta,$$

with at least one strict inequality, and

[7] Our focus here is on equity, not efficiency. In fact, in the model treated in this paper, if ex post utilities differ, $u_{h(\alpha)} \neq u_{h(\beta)}$, then the allocation is ex ante inoptimal (in the traditional sense).

$$\sum_{H} x'_h \leqq \sum_{H} x_h. \qquad \text{(ii)}$$

Compare our definition of dynamic Pareto optimality with the concept of conditional Pareto optimality introduced by Muench (1977). Also see Peled (forthcoming). Of course, an allocation is said to be Pareto optimal (in the traditional sense) if, in the definition above, the two conditions in (i) are replaced by the single condition

$$v_h(x'_h) \geqq v_h(x_h) \qquad \text{for } h \text{ in } H. \qquad \text{(i')}$$

Thus, every Pareto-optimal allocation is dynamically Pareto optimal, but the converse is not true.

The next two propositions are our formal welfare results. Market equilibria are identified with dynamically Pareto-optimal allocations. Nonsunspot equilibria are identified with Pareto-optimal allocations.

PROPOSITION 6: Every market equilibrium allocation is dynamically Pareto optimal. Every dynamically Pareto-optimal allocation can be achieved as a market equilibrium allocation under some assignment of endowments which may depend on sunspot activity (or, equivalently, under some lump-sum tax-subsidy scheme which may depend on sunspot activity).

PROOF: It follows directly from (7)–(9) and the traditional theorems of welfare economics (see, e.g., Arrow and Hahn 1971, chap. 4), when Mr. $h(\alpha)$ and Mr. $h(\beta)$ are treated as separate individuals if h is in G^1. If $h(\alpha)$ and $h(\beta)$ are to be treated separately, then obviously for the second part of the proposition, "their" endowments, $\omega_h(\alpha)$ and $\omega_h(\beta)$, will not in general be equal.

PROPOSITION 7: Every market equilibrium allocation in which sunspots do not matter is Pareto optimal. Every Pareto-optimal allocation can be achieved as a market equilibrium allocation (in which sunspots do not matter) under some assignment of endowments which is independent of sunspot activity (or, equivalently, under some lump-sum tax-subsidy scheme which is independent of sunspot activity).

PROOF: It follows from the equivalence of (traditional) certainty equilibria and nonsunspot equilibria—see proposition 4—together with the traditional theorems of welfare economics.

The Pareto-optimal allocations form a subset of the dynamically Pareto-optimal allocations. The Pareto-optimal allocations are, in fact, those dynamically Pareto-optimal allocations for which the planner has fixed the relative welfare weights between Mr. $h(\alpha)$ and Mr. $h(\beta)$ (for h in G^1) equal to the relative likelihoods of birth state α and birth state β. This may or may not be a socially desirable procedure.[8] If it is deemed worthy, though, it provides a justification for government intervention to offset the real effects of sunspots. Put another way, those who hold briefs for the traditional Pareto criterion as an ethical norm should tend to be especially sympathetic to government stabilization policies designed to counteract the effects of sunspots.

VIII. Concluding Remarks

In truly dynamic economies, there are inescapable restrictions on market participation. We have shown that the traditional notion of equilibrium in dynamic models is too narrow. Sunspots can matter.

How much do sunspots matter? Are sunspot equilibria "likely" relative to nonsunspot equilibria? Are they stable to shocks, including variations in government policies? We expect to be able to analyze these issues in the model developed in this paper. The model must be extended, however, in order to be able to analyze the macro-oriented questions raised by sunspot activity. We could then ask: Are sunspot equilibria more "likely" in monetary than in nonmonetary economies? What policies should the government pursue in the face of sunspot activity?[9]

Appendix

The special demographic structure underlying observations 1 and 2 allowed us to provide simple examples of market equilibria in which sunspots matter. In those particular examples, all sunspot equilibria are merely lotteries over certainty equilibria. Here we substantiate our claim that this result does not generalize.

For this purpose, we construct an economy in which there is a unique certainty equilibrium—and thus, by proposition 4, a unique nonsunspot equilibrium—yet in which there is at least one sunspot equilibrium. In other words, we establish that, in general, multiplicity of certainty equilibria is not required for there to be an equilibrium in which sunspots matter.[10] Our example, therefore, also illustrates the basic point that an equilibrium in which sunspots matter need not be based on a lottery over certainty equilibria.

Consider the general formulation which is the centerpiece of Section VI. If

[8] We tend to believe that this approach is not compelling.

[9] We have been discussing the "micro" questions with Yves Balasko. We expect the recent advances in overlapping-generations theory to be of use for the "macro" problems. We should record here that "sunspot activity" has also been observed to be emanating from Costas Azariadis, Roger Guesnerie, and Steve Spear.

[10] In our current research with Yves Balasko, we have completed the circle by demonstrating the converse result as well, namely, that the existence of sunspot equilibria is not necessary for there to be multiplicity of certainty equilibria. It is also noteworthy that the same analysis establishes that, generically, the sort of example we present here will have an even number of sunspot equilibria.

we are to go beyond observations 1 and 2, there must be at least two consumers in G^0. Hence, assume now that there are two consumers, Mr. 0_A and Mr. 0_B, born before sunspot activity is observed, and one consumer, Mr. 1, born afterward. These consumers will turn out to be very nicely behaved. As in the main body of the paper, utility functions satisfy standard assumptions (e.g., monotonicity and smoothness). Furthermore, demand functions are of the textbook variety (e.g., both commodities are normal goods for each consumer, and the first commodity is a gross substitute for the second). The reason for presenting the example here—rather than in Section VI—is that its construction is, of necessity, somewhat intricate.

In constructing this example, we adopt the following strategy: First, we specify an endowment vector which is independent of sunspot activity—that is, so that $\omega_h(s) = w_h$ for $h = 0_A$, 0_B, 1 and $s = \alpha$, β. Then we choose a contingent-claims price vector so that sunspots will matter—that is, so that $p(\alpha)$ is not proportional to $p(\beta)$—and a commodity allocation vector that is affordable (by each consumer) and feasible (for the economy), which satisfy the additional properties

$$x_{0_A}(\alpha) \not\leqq x_{0_A}(\beta),\ x_{0_B}(\alpha) > x_{0_B}(\beta) \text{ and } x_1(\alpha) < x_1(\beta) \tag{A1}$$

and

$$p(s) \cdot x_h(s) \neq p(s) \cdot w_h \tag{A2}$$

for $h = 0_A$, 0_B and $s = \alpha$, β. Because of (A1), it is then possible to find common probabilities $\pi(s)$ and additively separable utility functions

$$u_h(x^1, x^2) = \phi_h[f_h(x^1) + g_h(x^2)] \tag{A3}$$

such that the price vector is an equilibrium supporting the allocation vector. Thus, this is an equilibrium in which sunspots matter, while from (A2), prices q proportional to either $p(\alpha)$ or $p(\beta)$ will generally not be certainty equilibria.[11] Finally, when we limit the component functions g_h so that the derivative elasticity condition

$$-\frac{g_h''(x^2)x^2}{g_h'(x^2)} < \frac{x^2}{x_h^2 - w^2} \text{ for } x^2 \geqq w_h^2 \tag{A4}$$

is satisfied, it is further established that the construction is compatible with uniqueness of certainty equilibrium.

This program will be carried out in two steps. First, using the utility specification (A3), we exhibit an equilibrium p in which sunspots matter (and which will not generally reduce to a lottery over certainty equilibria). Second, using (for the first time) the elasticity condition (A4), we establish that this sort of equilibrium is consistent with there being only one certainty equilibrium q.

STEP 1: We begin by specifying endowments and designing prices and allocations which admit sufficient degrees of freedom to enable us then to select suitable probabilities and utility functions. The reader is urged to consult figure 5, which provides a geometric illustration of the method for designing the price-allocation pair. (i) Let $w_h = (1, 1)$ for $h = 0_A$, 0_B, and $w_1 =$

[11] That is, to put it another way, the sunspot equilibrium p will generally not be a lottery over certainty equilibria $q^\alpha = p(\alpha)/\pi(\alpha)$ and $q^\beta = p(\beta)/\pi(\beta)$. This interesting result by itself can be guaranteed by imposing additional, mild restrictions on, say, the component functions g_{0_A} and g_{0_B}. But such restrictions are superfluous in view of the much stronger result which we state next.

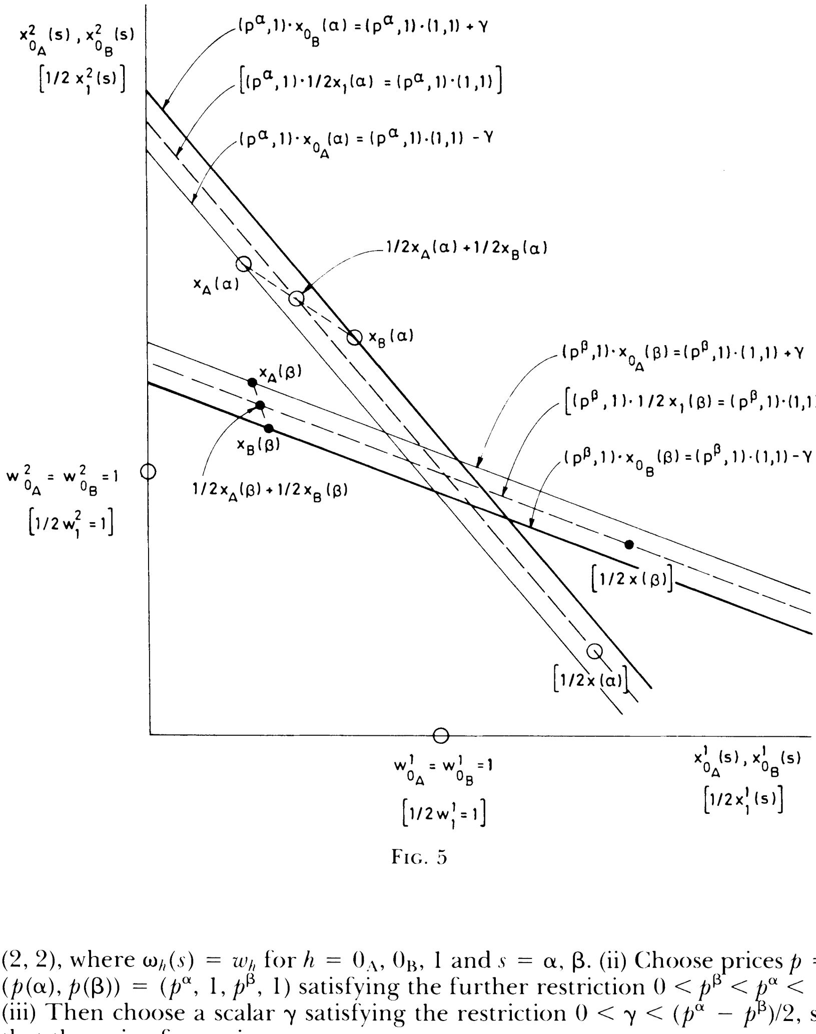


FIG. 5

(2, 2), where $\omega_h(s) = w_h$ for $h = 0_A, 0_B, 1$ and $s = \alpha, \beta$. (ii) Choose prices $p = (p(\alpha), p(\beta)) = (p^\alpha, 1, p^\beta, 1)$ satisfying the further restriction $0 < p^\beta < p^\alpha < 1$. (iii) Then choose a scalar γ satisfying the restriction $0 < \gamma < (p^\alpha - p^\beta)/2$, so that the pair of equations

$$(p^\alpha, 1)\cdot(x^1, x^2) = (p^\alpha, 1)\cdot(1, 1) - \gamma$$

and

$$(p^\beta, 1)\cdot(x^1, x^2) = (p^\beta, 1)\cdot(1, 1) + \gamma$$

has a positive solution (x^1, x^2).

Consider $x_{0_A} = x_A = (x_A(\alpha), x_A(\beta)) = (\xi_A, \theta_A\xi_A, p^\alpha(1 - \xi_A) + (1 - \gamma), p^\beta(1 - \theta_A\xi_A) + (1 + \gamma))$ such that $0 < \xi_A < 1$, $1 < \theta_A < 1/\xi_A$, and $p^\alpha(1 - \xi_A) + (1 - \gamma) > p^\beta(1 - \theta_A\xi_A) + (1 + \gamma)$. In other words, consider choosing $x_{0_A} = x_A$ so that

$$(p^\alpha, 1)\cdot x_A(\alpha) = (p^\alpha, 1)\cdot(1, 1) - \gamma$$

and

$$(p^{\beta}, 1) \cdot x_A(\beta) = (p^{\beta}, 1) \cdot (1, 1) + \gamma,$$

which implies that $(p^{\alpha}, 1) \cdot x_A(\alpha) + (p^{\beta}, 1) \cdot x_A(\beta) = (p^{\alpha}, 1) \cdot (1, 1) + (p^{\beta}, 1) \cdot (1, 1)$, and

$$0 < x_A^1(\alpha) < x_A^1(\beta) < 1$$

and

$$2 > x_A^2(\alpha) > x_A^2(\beta) > 1.$$

Similarly, consider $x_{0_B} = x_B = (x_B(\alpha), x_B(\beta)) = (\xi_B, \theta_B\xi_B, p^{\alpha}(1 - \xi_B) + (1 + \gamma), p^{\beta}(1 - \theta_B\xi_B) + (1 - \gamma))$ such that $0 < \xi_B < 1$, $0 < \theta_B < 1$, and $p^{\alpha}(1 - \xi_B) + (1 + \gamma) > p^{\beta}(1 - \theta_B\xi_B) + (1 - \gamma)$—or choosing $x_{0_B} = x_B$ so that

$$(p^{\alpha}, 1) \cdot x_B(\alpha) = (p^{\alpha}, 1) \cdot (1, 1) + \gamma$$

and

$$(p^{\beta}, 1) \cdot x_B(\beta) = (p^{\beta}, 1) \cdot (1, 1) - \gamma,$$

which implies that $(p^{\alpha}, 1) \cdot x_B(\alpha) + (p^{\beta}, 1) \cdot x_B(\beta) = (p^{\alpha}, 1) \cdot (1, 1) + (p^{\beta}, 1) \cdot (1, 1)$, and

$$1 > x_B^1(\alpha) > x_B^1(\beta) > 0$$

and

$$2 > x_B^2(\alpha) > x_B^2(\beta) > 1.$$

(The representation of x_h in terms of the parameters ξ_h and θ_h turns out to be useful in subsequent calculations. Also, note that a whole subscript, e.g., 0_A, is used in denoting a variable, while a partial subscript, e.g., A, is used in denoting a particular value of that variable. Thus, x_A is a particular value of the variable x_{0_A}, x is a particular value of the variable x_1, and so forth.)

(iv) Choose ξ_A, θ_A, ξ_B, and θ_B so that, in addition, $\gamma \leqq (1 - \theta_A\xi_A)(p^{\alpha} - p^{\beta})/2$ and $\xi_A(\theta_A - 1) < \xi_B(\theta_B - 1)$. In particular, the second of these restrictions implies that $[1 - x_A^1(\alpha)] + [1 - x_B^1(\alpha)] < [1 - x_A^1(\beta)] + [1 - x_B^1(\beta)]$, while the previous restrictions, $x_A^2(\alpha) > x_A^2(\beta)$ and $x_B^2(\alpha) > x_B^2(\beta)$, imply that $[1 - x_A^2(\alpha)] + [1 - x_B^2(\alpha)] < [1 - x_A^2(\beta)] + [1 - x_B^2(\beta)]$. The role of the first restriction will be explained later.

It is easily verified that all the various restrictions on ξ_h and θ_h (or, equivalently, x_A and x_B) will obtain, for instance, if $\xi_A \sim 0$, $\xi_B = \frac{1}{2}$, and $\theta_A - 1 = 1 - \theta_B \sim 0$.

(v) Finally, choose $x_1 = x = (x(\alpha), x(\beta))$ so that $x^i(s) = [1 - x_A^i(s)] + [1 - x_B^i(s)] + 2$ for $i = 1, 2$; $s = \alpha, \beta$, which, from the foregoing, implies that $(p^s, 1) \cdot x(s) = (p^s, 1) \cdot (2, 2)$ for $s = \alpha, \beta$, $0 < x(\alpha) < x(\beta)$, and $x_A^i(s) + x_B^i(s) + x_B^i(s) = 1 + 1 + 2$ for $i = 1, 2$; $s = \alpha, \beta$.

When one refers to figure 5, it should be clear that, given the price vector $(p^{\alpha}, 1, p^{\beta}, 1)$, the allocation vector $(x_A(\alpha), x_A(\beta), x_B(\alpha), x_B(\beta), x(\alpha), x(\beta))$ satisfies the budget and nonnegativity constraints in (7) and (8), as well as the materials balance condition (9). Thus, for the first step of our program, all that remains is to verify that with judicious choices of probabilities and utility functions, the allocation vector also yields optimal solutions to the programming problems (7) and (8).

Consider u_h having the additively separable form (A3) with ϕ_h: $\mathbb{R} \rightarrow \mathbb{R}$, f_h:

$\mathbb{R}_+ \to \mathbb{R}$ and g_h: $\mathbb{R}_+ \to \mathbb{R}$ (at least) twice continuously differentiable, strictly increasing, and (at least for one of the functions, strictly) concave. Our argument will rely heavily on the elementary result that, given constants $0 < \xi_1 < \xi_2$ (respectively, $\xi_1 < \xi_2$) and $0 < \lambda < 1$ (respectively, $0 < \lambda \leqq 1$), one can always find a function Ψ: $\mathbb{R}_+ \to \mathbb{R}$ (respectively, Ψ: $\mathbb{R} \to \mathbb{R}$) which is smooth, strictly increasing, and strictly concave (respectively, concave) and which satisfies the conditions

$$\lim_{\xi \to 0^+} \Psi'(\xi) = \infty \text{ and } \frac{\Psi'(\xi_2)}{\Psi'(\xi_1)} = \lambda, \text{ respectively } \frac{\Psi'(\xi_2)}{\Psi'(\xi_1)} = \lambda.$$

In particular, Ψ can always be chosen from the class of functions having constant derivative elasticity, so that $\Psi'(\xi) = M\xi^{-\mu}$ for $\xi \geqq 0$ with $M > 0$ and $\mu > 0$. In using this simple observation to construct utility functions, we begin with Mr. 1, proceed to Mr. 0_B, and finish with Mr. 0_A, since their respective consumption bundles are increasingly more complicated to reconcile with optimizing behavior.

So consider Mr. 1, and suppose that $u_1(x^1, x^2) = \phi_1[f_1(x^1) + g_1(x^2)]$. Then direct application of the Kuhn-Tucker theorem for concave programming (together with some straightforward manipulation) yields the result that x solves the problem

maximize

$$\phi_1\{f_1[x_1^1(s)] + g_1[x_1^2(s)]\}$$

subject to

$$(p^s, 1) \cdot x_1(s) = (p^s, 1) \cdot (2, 2)$$

and

$$x_1(s) \geqq 0$$

for $s = \alpha, \beta$ if and only if x satisfies the first-order conditions

$$\frac{f_1'[x^1(\alpha)]}{g_1'[x^2(\alpha)]} = p^\alpha$$

and

$$\frac{f_1'[x^1(\alpha)]}{f_1'[x^2(\beta)]} = \frac{p^\alpha g_1'[x^2(\alpha)]}{p^\beta g_1'[x^2(\beta)]}. \tag{A5}$$

But since, by construction, $p^\alpha/p^\beta > 1$ while $x(\alpha) < x(\beta)$, these conditions will obtain if we choose $\phi_1' = c > 0$, and $f_1 = f$ and $g_1 = g$, so that

$$\frac{f'[x^1(\alpha)]}{g'[x^2(\alpha)]} = p^\alpha, \; \frac{g'[x^2(\alpha)]}{g'[x^2(\beta)]} = \frac{1}{\lambda} > 1, \text{ and } \frac{f'[x^1(\alpha)]}{f'[x^1(\beta)]} = \frac{p^\alpha}{p^\beta}\frac{1}{\lambda} > 1. \tag{A6}$$

An example of such functions is depicted in figure 6.

For Mr. 0_B, similar reasoning yields the result that x_B solves the problem

maximize

$$\pi(\alpha)\phi_{0_B}\{f_{0_B}[x_{0_B}^1(\alpha)] + g_{0_B}[x_{0_B}^2(\alpha)]\} + \pi(\beta)\phi_{0_B}\{f_{0_B}[x_{0_B}^1(\beta)] + g_{0_B}[x_{0_B}^2(\beta)]\}$$

subject to

$$(p^\alpha, 1) \cdot x_{0_B}(\alpha) + (p^\beta, 1) \cdot x_{0_B}(\beta) \leqq (p^\alpha, 1) \cdot (1, 1) + (p^\beta, 1) \cdot (1, 1)$$

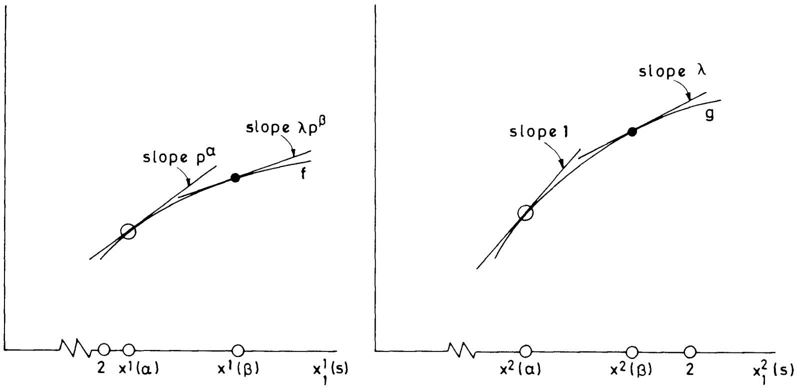


FIG. 6

and

$$x_{0_B} \geqq 0$$

if and only if x_B satisfies the first-order conditions

$$\frac{f'_{0_B}[x^1_B(\alpha)]}{g'_{0_B}[x^2_B(\alpha)]} = p^{\alpha},$$

$$\frac{f'_{0_B}[x^1_B(\alpha)]}{f'_{0_B}[x^1_B(\beta)]} = \frac{p^{\alpha} g'_{0_B}[x^2_B(\alpha)]}{p^{\beta} g'_{0_B}[x^2_B(\beta)]}, \tag{A7}$$

and

$$\frac{\phi'_{0_B}\{f_{0_B}[x^1_B(\alpha)] + g_{0_B}[x^2_B(\alpha)]\}}{\phi'_{0_B}\{f_{0_B}[x^1_B(\beta)] + g_{0_B}[x^2_B(\beta)]\}} \frac{g'_{0_B}[x^2_B(\alpha)]}{g'_{0_B}[x^2_B(\beta)]} = \frac{\pi(\beta)}{\pi(\alpha)}.$$

In this case, since by construction $x_B(\alpha) > x_B(\beta)$, these conditions will obtain if we choose $\pi(\beta)/\pi(\alpha) = [1 - \pi(\alpha)]/\pi(\alpha) < p^{\beta}/p^{\alpha}$ and $\phi'_{0_B} = c_B > 0$, and $f_{0_B} = f_B$ and $g_{0_B} = g_0$, so that

$$\frac{f'_B[x^1_B(\alpha)]}{g'_B[x^2_B(\alpha)]} = p^{\alpha},$$

$$\frac{g'_B[x^2_B(\alpha)]}{g'_B[x^2_B(\beta)]} = \frac{\pi(\beta)}{\pi(\alpha)} < \frac{p^{\beta}}{p^{\alpha}}, \tag{A8}$$

and

$$\frac{f'_B[x^1_B(\alpha)]}{f'_B[x^1_B(\beta)]} = \frac{p^{\alpha}}{p^{\beta}} \frac{\pi(\beta)}{\pi(\alpha)} < 1.$$

Finally, consider Mr. 0_A. From an argument parallel with that for Mr. 0_B, we know that x_A must satisfy first-order conditions like (A7),

$$\frac{f'_{0_\Lambda}[x^1_\Lambda(\alpha)]}{g'_{0_\Lambda}[x^2_\Lambda(\alpha)]} = p^\alpha,$$

$$\frac{f'_{0_\Lambda}[x^1_\Lambda(\alpha)]}{f'_{0_\Lambda}[x^1_\Lambda(\beta)]} = \frac{p^\alpha g'_{0_\Lambda}[x^2_\Lambda(\alpha)]}{p^\beta g'_{0_\Lambda}[x^2_\Lambda(\beta)]}, \tag{A9}$$

and

$$\frac{\phi'_{0_\Lambda}\{f_{0_\Lambda}[x^1_\Lambda(\alpha)] + g_{0_\Lambda}[x^2_\Lambda(\alpha)]\}}{\phi'_{0_\Lambda}\{f_{0_\Lambda}[x^1_\Lambda(\beta)] + g_{0_\Lambda}[x^2_\Lambda(\beta)]\}} \frac{g'_{0_\Lambda}[x^2_\Lambda(\alpha)]}{g'_{0_\Lambda}[x^2_\Lambda(\beta)]} = \frac{\pi(\beta)}{\pi(\alpha)}.$$

The difficulty here is that we have just chosen $\pi(\beta)/\pi(\alpha) < p^\beta/p^\alpha$, while, by earlier construction, $x^1_\Lambda(\alpha) < x^1_\Lambda(\beta)$ and $x^2_\Lambda(\alpha) > x^2_\Lambda(\beta)$. However, this is where we can employ the extra degree of freedom afforded by cleverly selecting the transforming function ϕ_{0_Λ} (representing "overall" risk aversion). For, if we knew that

$$f_{0_\Lambda}[x^1_\Lambda(\alpha)] + g_{0_\Lambda}[x^2_\Lambda(\alpha)] > f_{0_\Lambda}[x^1_\Lambda(\beta)] + g_{0_\Lambda}[x^2_\Lambda(\beta)], \tag{A10}$$

then we could choose $\phi_{0_\Lambda} = \phi_\Lambda$, $f_{0_\Lambda} = f_\Lambda$, and $g_{0_\Lambda} = g_\Lambda$ so that

$$\frac{f'_\Lambda[x^1_\Lambda(\alpha)]}{g'_\Lambda[x^2_\Lambda(\alpha)]} = p^\alpha,\ \frac{p^\beta}{p^\alpha} < \frac{g'_\Lambda[x^2_\Lambda(\alpha)]}{g'_\Lambda[x^2_\Lambda(\beta)]} = \lambda < 1,$$

$$\frac{f'_\Lambda[x^1_\Lambda(\alpha)]}{f'_\Lambda[x^1_\Lambda(\beta)]} = \frac{p^\alpha}{p^\beta}\lambda > 1, \tag{A11}$$

and

$$\frac{\phi'_\Lambda\{f_\Lambda[x^1_\Lambda(\alpha)] + g_\Lambda[x^2_\Lambda(\alpha)]\}}{\phi'_\Lambda\{f_\Lambda[x^2_\Lambda(\beta)] + g_\Lambda[x^2_\Lambda(\beta)]\}}\lambda = \frac{\pi(\beta)}{\pi(\alpha)}.$$

And our earlier limitation on the choice of x_Λ that $\gamma \leqq (1 - \theta_\Lambda\xi_\Lambda)(p^\alpha - p^\beta)/2$ was designed precisely to ensure (A10): To begin with, notice that the first listed condition in (A9) is essentially a normalization of f'_{0_Λ}, in terms of $p^\alpha g'_{0_\Lambda}[x^2_\Lambda(\alpha)]$, and is therefore only tangentially related to the other two conditions. If we utilize this observation together with the representation of x_Λ in terms of ξ_Λ and θ_Λ, it follows that (A10) holds (for smooth and strictly concave functions f_{0_Λ} and g_{0_Λ} restricted only by the first listed condition in [A9]) if

$$f_{0_\Lambda}[x^1_\Lambda(\beta)] - f_{0_\Lambda}[x^1_\Lambda(\alpha)] < f'_{0_\Lambda}[x^1_\Lambda(\alpha)](x^1_\Lambda(\beta) - x^1_\Lambda(\alpha))$$
$$\leqq g'_{0_\Lambda}[x^2_\Lambda(\alpha)](x^2_\Lambda(\alpha) - x^2_\Lambda(\beta)) < g_{0_\Lambda}[x^2_\Lambda(\alpha)] - g_{0_\Lambda}[x^2_\Lambda(\beta)]$$

if

$$p^\alpha(x^1_\Lambda(\beta) - x^1_\Lambda(\alpha)) \leqq x^2_\Lambda(\alpha) - x^2_\Lambda(\beta)$$

if

$$p^\alpha(\theta_\Lambda\xi_\Lambda - \xi_\Lambda) \leqq [p^\alpha(1 - \xi_\Lambda) + (1 - \gamma)] - [p^\beta(1 - \theta_\Lambda\xi_\Lambda) + (1 + \gamma)]$$

if

$$\gamma \leqq (1 - \theta_\Lambda\xi_\Lambda)(p^\alpha - p^\beta)/2.$$

This completes the first step of our program.

STEP 2: It remains only to demonstrate that the procedure of step 1 can also be carried out under the additional restriction that the component functions g_h satisfy the elasticity condition (A4) and that, for such an economy, there is a unique certainty equilibrium. It turns out that we have enough flexibility even after further limiting attention to component functions exhibiting constant elasticity, that is, g_h satisfying

$$g_h'(x^2) = N(x^2)^{-\nu} \text{ for } x^2 \geqq 0 \text{ with } N > 0 \text{ and } 0 < \nu \leqq 1. \qquad (A4')$$

The essential point is to ensure that this strengthened condition, (A4′), is consistent with (A6), (A8), and (A11), since, given the original condition, (A4), a routine argument (which we outline below for the sake of completeness) establishes uniqueness of certainty equilibrium.

So, once more, consider the optimizing behavior of Mr. 1, Mr. 0_B, and Mr. 0_A, in turn, as described in (A6), (A8), and (A11), respectively. It is immediately obvious that there is no problem with Mr. 1, since

$$\frac{g'[x^2(\alpha)]}{g'[x^2(\beta)]} = \frac{N[x^2(\alpha)]^{-\nu}}{N[x^2(\beta)]^{-\nu}} = \left[\frac{x^2(\beta)}{x^2(\alpha)}\right]^{\nu} > 1$$

for every ($N > 0$ and) $0 < \nu \leqq 1$. For Mr. 0_B and Mr. 0_A, the argument is only slightly more complicated; we merely need to verify that, given all previous restrictions on x_B and x_A, we can still choose these consumption bundles so that

$$\frac{g_B'[x_B^2(\alpha)]}{g_B'[x^2(\beta)]} = \left[\frac{x_B^2(\beta)}{x_B^2(\alpha)}\right]^{\nu_B} < \frac{p^\beta}{p^\alpha} < \frac{g_A'[x_A^2(\alpha)]}{g_A'[x^2(\beta)]} = \left[\frac{x_A^2(\beta)}{x_A^2(\alpha)}\right]^{\nu_A} < 1 \qquad (A12)$$

for some $0 < \nu_B \leqq 1$ and $0 < \nu_A \leqq 1$. In fact, it is easily demonstrated that $\nu_B = \nu_A = 1$ will do: again, if we explicitly use the representation of x_h in terms of ξ_h and θ_h, then (A12) becomes

$$\frac{p^\beta(1 - \theta_B\xi_B) + (1 - \gamma)}{p^\alpha(1 - \xi_B) + (1 + \gamma)} < \frac{p^\beta}{p^\alpha} < \frac{p^\beta(1 - \theta_A\xi_A) + (1 + \gamma)}{p^\alpha(1 - \xi_A) + (1 - \gamma)} < 1. \qquad (A13)$$

But, just as before, (A13) will obtain, for instance, if $\xi_B = \frac{1}{2}$, $\xi_A \sim 0$, and $1 - \theta_B = \theta_A - 1 \sim 0$.

In order to show that for this sort of economy there can be only a single certainty equilibrium, it suffices to show that aggregate excess demand for commodity 1 is strictly decreasing in the relative price of commodity 1 (when all consumers ignore sunspot activity). For this purpose, it is unnecessary to distinguish between the three consumers since each has the same type of utility function and, hence, the same structure of excess demand. Thus, following the lead from the last part of Section IV, we now let prices of commodities be (q^1, q^2), the representative consumer's demand for commodities be (y^1, y^2), his endowment be (w^1, w^2), his excess demand be $(z^1, z^2) = (y^1 - w^1, y^2 - w^2)$ and—when needed later—his income be $w = q^1w^1 + q^1w^2$. Also, suppose that commodity 2 is the numeraire, so that $q^2 = 1$ and $q^1/q^2 = q^1$. Then it is straightforward to derive the following excess demand function for commodity 1 given that the representative consumer's utility function is additively separable, $u(y^1, y^2) = \phi[f(y^1) + g(y^2)]$:[12]

[12] The calculation assumes that

$$\lim_{y^1 \to 0^+} f'(y^1) = \lim_{y^2 \to 0^+} g'(y^2) = \infty,$$

which is consistent with our preceding construction. Notice also that the transforming function ϕ plays no role here (though it clearly did before); if there is no risk, then

$$f'(w^1 + z^1) - g'(w^2 - q^1 z^1)q^1 = 0 \text{ for } q^1 > 0. \tag{A14}$$

Implicit differentiation of (A14) yields, after some simplification (in particular, when the budget constraint $-q^1 z^1 = z^2$ is used),

$$\begin{aligned}\frac{dz^1}{dq^1} &= \frac{g'(w^2 - q^1 z^1) - g''(w^2 - q^1 z^1)q^1 z^1}{f''(w^1 + z^1) + g''(w^2 - q^1 z^1)(q^1)^2} \\ &= \frac{g'(w^2 + z^2)}{f''(w^1 + z^1) + g''(w^2 + z^2)(q^1)^2}\left[1 + \frac{g''(w^2 + z^2)z^2}{g'(w^2 + z^2)}\right] \\ &< 0\end{aligned}$$

if and only if

$$1 + \frac{g''(w^2 + z^2)z^2}{g'(w^2 + z^2)} > 0$$

or

$$-\frac{g''(y^2)y^2}{g'(y^2)} < \frac{y^2}{y^2 - w^2} \text{ for } y^2 \geqq w^2.$$

That is, under the elasticity condition (A4), each consumer's excess demand (and hence the economy's aggregate excess demand) for commodity 1 must be strictly decreasing in that commodity's relative price.

This completes the second and final step of our program. We have constructed an economy with only one certainty equilibrium and at least one sunspot equilibrium.

Our final calculation is intended to underline the fact that there is nothing unusual about the specification (A3)–(A4). Essentially repeating the immediately preceding analysis, but now in terms of the ordinary demand function for commodity 1 (depending on the relative price q^1 and correspondingly measured income w), reveals the following property of the Engle curves for the representative consumer:

$$\frac{\partial y^1}{\partial w} = \frac{g''(y^2)q^1}{f''(y^1) + g''(y^2)(q^1)^2} > 0$$

and

$$\frac{\partial y^2}{\partial w} = 1 - q^1\frac{\partial y^1}{\partial w} = 1 - \frac{g''(y^2)(q^1)^2}{f''(y^1) + g''(y^2)(q^1)^2} > 0.$$

That is, both commodities are normal goods for each consumer.

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overall aversion to risk has absolutely no effect on individual behavior. This result is of critical importance to the approach we have taken in this Appendix, since it introduces a significant wedge between the economy with and the economy without extrinsic uncertainty.

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