Equilibrium analysis

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CHAPTER 5

Lump-sum taxes and transfers: public debt in the overlapping-generations model

Yves Balasko and Karl Shell

1 Introduction, summary, and reader's guide

Americans are hearing a lot these days about the government deficit and its integral, the public debt. The subject is hardly new for students of public finance and macroeconomics. In this chapter, we return to square one. We examine the nature of the intertemporal consistency restrictions imposed on the government's fiscal policy. In particular, we evaluate the basis for the current ("neo-Ricardian") fixation with long-run debt retirement.

Our analysis is cast in terms of the standard overlapping-generations model of exchange. Consumers are assumed to possess perfect foresight. The government commits itself to a full intertemporal fiscal policy, which it announces at the beginning of time. There is assumed to be neither intrinsic uncertainty nor extrinsic uncertainty. The spot markets and the borrowing-and-lending markets are complete and competitive. Participation in these markets is restricted only by the natural lifetimes of the consumers. Since sunspot equilibria are ruled out by assumption, the restrictions on market participation are not essential.

Trading is assumed to be costless. Financial assets have only two roles. They are potential value stores. They can also be used by consumers in paying their taxes (and by the government in distributing transfers). In this environment, financial instruments are in essence identical. We can assume, therefore, that there is only one type of instrument, called "money."

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(Since it serves no special role in facilitating exchange, you might prefer to think of this instrument as a "bond," a bond that does not pay nominal interest but can appreciate in value relative to commodities. The absence of nominal interest is a pure technicality; nothing of substance is lost by its exclusion.)

It is assumed for simplicity that the government makes no expenditures on goods and services. The government is assumed to have the power to levy costless lump-sum taxes (and distribute costless lump-sum transfers) *in terms of money*. In this respect, that is, in couching the analysis in terms of nominal taxes rather than in terms of real taxes, the present study is a companion to our analysis of money taxation in the static economy (Balasko and Shell 1983). Recall that a tax-transfer policy is said to be *bonafide* (for a given economy) if it is consistent with some competitive equilibrium in which the goods price of money is nonzero (i.e., in which the general price level is finite). If money is a free good, then the money tax is no burden to the taxed consumer (nor is the money transfer any benefit to the subsidized consumer). Hence, if a fiscal policy is not bonafide, it cannot possibly affect the allocation of resources.

In the static economy, a tax-transfer (or fiscal) policy is said to be *balanced* if the algebraic sum of the transfers is zero. We have established elsewhere (Balasko and Shell 1983) that for the static economy the government fiscal policy is bonafide if and only if it is balanced. In this chapter, we extend this result to the finite-horizon, overlapping-generations economy. Balanced policies are then interpreted as those having the propery that the public debt is retired at the terminal date. If the world is known to end with certainty on a given date, the following are true: (1) If the debt is not perfectly retired on the terminal date, then money must be worthless at each date, and (2) if the debt is perfectly retired on the terminal date, then there is a range of positive goods prices of money consistent with competitive equilibrium.

We use our results for finite-horizon economies to establish our basic theorem for infinite-horizon, overlapping-generations economies: *Strongly balanced tax-transfer policies,* policies for which the public debt is forever zero after some finite date, *are always bonafide.* It does not appear to be possible to strengthen this theorem in any interesting way. Without restricting preferences and endowments, weaker notions of balancedness do not necessarily imply bonafidelity. Hence, if the public debt is retired by some finite date, after which time the government neither taxes nor makes transfers, then there are positive goods prices of money consistent with competitive equilibrium. Nonetheless, even if the public debt vanishes asymptotically, money is necessarily worthless if debt retirement is too slow for the given economy.

Furthermore, the government can select a fiscal policy for which the debt is not retired in any sense, yet there can be positive goods prices of money consistent with competitive equilibrium. Indeed, it is well known that infinite-horizon economies can be constructed so that the obviously nonbalanced, constant-public-debt policy is bonafide.

The overlapping-generations model is described in Section 2. We employ notation that allows us to present the finite-horizon version and the infinite-horizon version simultaneously. The infinite-horizon version reduces to the model used in Balasko and Shell (1981).

The finite-horizon version of the overlapping-generations model is analyzed in Section 3. The identity between the set of bonafide tax-transfer policies and the set of balanced tax-transfer policies is established through an analysis of an associated no-taxation economy. The set of tax-transfer policies consistent with equilibrium in which the price of money is 1 is shown to be bounded and arc connected and to contain zero in its relative interior. There is a continuum of equilibria for each balanced tax-transfer policy. In the set of equilibrium money prices, zero is not isolated.

The infinite-horizon version of the overlapping-generations model is analyzed in Sections 4 and 5. This analysis can be taken as an extension of our earlier work (Balasko and Shell 1981). In Section 4, we use the central result of Section 3 to show that strongly balanced tax-transfer policies are bonafide. In Section 5, we construct two examples. The first example establishes that asymptotically balanced tax-transfer policies are not necessarily bonafide. The second example establishes that recurrently balanced tax-transfer policies, policies for which the public debt is zero infinitely often, are not necessarily bonafide. Even recurrently balanced policies with the further property that the *limit superior* of the public debt vanishes are not necessarily bonafide.

Two important points emerge from our analysis. First, it matters if taxes are set in money terms rather than in real terms. Even in the case of perfect lump-sum *money* taxation, the government does not have full fiscal potency because it cannot determine the general price level. Second, it matters if the economic horizon is essentially infinite rather than being based on some known finite date. For the finite-horizon case, we know that a fiscal policy is bonafide if and only if the debt is perfectly retired on the terminal date. For the infinite-horizon case, there are examples of economies and fiscal policies for which there is asymptotic debt retirement but not bonafidelity, and there are cases without debt retirement that, nonetheless, exhibit bonafidelity.

2 The overlapping-generations model with lump-sum taxes and transfers

We summarize here some essential elements of the overlapping-generations model. Our model is based on that of Balasko and Shell (1981), but there is one very important difference: Here we allow for the possibility of a finite-time horizon as well as an infinite horizon.

We postulate a very simple demographic pattern, but this causes no loss in generality. Consumer h $(h=0,1,\ldots,T)$ is indexed by his place in the birth order. He is either present at the beginning of the economy $(h=0)$ and lives out the balance of his life in period 1 or he is born in the beginning of period t $(h=t;\ t=1,\ldots,T)$ and lives out his life in periods t and $t+1$. For the finite-horizon variant of the model, T is a (finite) positive integer. For the infinite-horizon variant, we take $T=+\infty$ in a way that makes the model identical to that presented in Balasko and Shell (1981).

In each period t $(t=1,\ldots,T+1)$, there are ℓ perishable commodities. The government creates and destroys otherwise imperishable fiat money. There is no other production. Let $x_t^{s,i}$ be the consumption of commodity i $(i=1,\ldots,\ell)$ by consumer t in period s. The preferences of consumer t are described by the utility function

$$u_t(x_t) \quad \text{for } t=0,1,\ldots,T,$$

where

$$x_0=x_0^1=(x_0^{1,1},\ldots,x_0^{1,\ell})\in \mathbb{R}_{++}^{\ell}$$

and

$$x_t=(x_t^t,x_t^{t+1})=(x_t^{t,1},\ldots,x_t^{t,\ell},x_t^{t+1,1},\ldots,x_t^{t+1,\ell})\in \mathbb{R}_{++}^{2\ell} \quad \text{for } t=1,\ldots,T.$$

Also denote by x_0 and x_t the respective vectors in $(\mathbb{R}_{++}^{\ell})^{T+1}$, $x_0=(x_0^1,0,\ldots,0)$ and $x_t=(0,\ldots,0,x_t^t,x_t^{t+1},0,\ldots,0)$ for $t=1,\ldots,T$. [If $T=\infty$, then under this latter interpretation we have, for example, that x_t is the sequence $(0,\ldots,0,x_t^t,x_t^{t+1},0,\ldots)$ in $(\mathbb{R}_{++}^{\ell})^{\infty}$. See Balasko and Shell (1981), especially Section 2.] Let $x=(x_0,x_1,\ldots,x_T)$ be the commodity allocation vector and $X=\mathbb{R}_{++}^{\ell(2T+1)}$ be the commodity space. The utility function u_0 (respectively u_t, $t=1,\ldots,T$) is assumed to be strictly increasing, smooth, and strictly quasiconcave on $\mathbb{R}_{++}^{\ell}$ (respectively, $\mathbb{R}_{++}^{2\ell}$). To avoid messy boundary problems, we also assume that the closure of each indifference surface in $\mathbb{R}^{\ell}$ (respectively, $\mathbb{R}^{2\ell}$) is contained in $\mathbb{R}_{++}^{\ell}$ (respectively, $\mathbb{R}_{++}^{2\ell}$).

Commodity endowments are given by

$$\omega_0 = \omega_0^1 = (\omega_0^{1,1}, \dots, \omega_0^{1,\ell}) \in \mathbb{R}^{\ell}_{++}$$

and

$$\omega_t = (\omega_t^t, \omega_t^{t+1}) = (\omega_t^{t,1}, \dots, \omega_t^{t,\ell}, \omega_t^{t+1,1}, \dots, \omega_t^{t+1,\ell}) \in \mathbb{R}^{2\ell}_{++} \quad \text{for } t = 1, \dots, T.$$

Also denote by ω_0 and ω_t the respective vectors in $(\mathbb{R}^{\ell}_{++})^{T+1}$, $\omega_0 = (\omega_0^1, 0, \dots, 0, \dots)$ and $\omega_t = (0, \dots, 0, \omega_t^t, \omega_t^{t+1}, 0, \dots, 0)$ for $t = 1, \dots, T$. Let $\omega = (\omega_0, \omega_1, \dots, \omega_T) \in X$ denote the commodity endowment vector.

Let $\mu_t \in \mathbb{R}$ be the nominal (money) lump-sum transfer to consumer t $(t = 0, 1, \dots, T)$. Let the tax-transfer vector be denoted by

$$\mu = (\mu_0, \mu_1, \dots, \mu_t, \dots, \mu_T) \in \mathbb{R}^{T+1} = \mathfrak{M},$$

the space of lump-sum tax-transfer policies.

Let $p^{t,i} \in \mathbb{R}_{++}$ be the price of commodity i $(i = 1, \dots, \ell)$ in period t, and define $p^t \in \mathbb{R}^{\ell}_{++}$ by $p^t = (p^{t,1}, \dots, p^{t,i}, \dots, p^{t,\ell})$. Then define the commodity price vector $p \in \mathbb{R}^{\ell(T+1)}_{++}$ by $p = (p^1, \dots, p^t, \dots, p^{T+1})$. Let $p^m \in \mathbb{R}_+$ be the price of money, so that the wealth of consumer t is given by

$$w_t = p \cdot \omega_t + p^m \mu_t$$

for $t = 0, 1, \dots, T$. Let $w = (w_0, w_1, \dots, w_T) \in W = \mathbb{R}^{T+1}_{++}$ be the vector of consumer incomes (or, more accurately, consumer wealths). Let $\mathcal{P}$ be the set of present prices, $\mathcal{P} = \{p \in (\mathbb{R}^{\ell}_{++})^{T+1} \mid p^{1,1} = 1\}$. Define the demand functions $f_t : (p, w_t) \mapsto x_t$ for $t = 0, 1, \dots, T$. Thus, we have $f_t : \mathcal{P} \times \mathbb{R}_{++} \to (\mathbb{R}^{\ell}_{++})^{T+1}$ for $t = 0, 1, \dots, T$.[1]

2.1. Definition. For the given commodity endowments $\omega \in X$ and the given tax transfer policy $\mu \in \mathfrak{M}$, a *competitive equilibrium* is a price system $(p, p^m) \in \mathcal{P} \times \mathbb{R}_+$ that solves the equations

$$\sum_{t=0}^{T} f_t(p, w_t) = \sum_{t=0}^{T} \omega_t = r$$

and

$$w_t = p \cdot \omega_t + p^m \mu_t \quad \text{for } t \geq 0,$$

where $r = (r^1, \dots, r^t, \dots, r^{T+1}) \in (\mathbb{R}^{\ell}_{++})^{T+1}$ is the vector of aggregate resources.

Let $q = (p, p^m)$ and $\mathcal{Q} = \mathcal{P} \times \mathbb{R}_+$.

2.2. Definition. Let $\mathcal{Q}(\omega, \mu)$ denote the *set of competitive equlibria* for $\omega \in X$ and $\mu \in \mathfrak{M}$. Thus, we have $\mathcal{Q}(\omega, \mu) = \{q \in \mathcal{Q} \mid \text{Definition 2.1 is satisfied for } \omega \in X \text{ and } \mu \in \mathfrak{M}\}$. The price system $q = (p, p^m)$ is said to define a *proper monetary equilibrium* for $\omega \in X$ and $\mu \in \mathfrak{M}$ if we have $q = (p, p^m) \in \mathcal{Q}(\omega, \mu)$ and $p^m > 0$.

2.3. Proposition. For each positive scalar λ, we have

$$\mathcal{Q}(\omega, \lambda\mu) = \{(p, p^m/\lambda) \mid (p, p^m) \in \mathcal{Q}(\omega, \mu)\}.$$

Proof: From the definition of wealth, we have $w_t = p \cdot \omega_t + p^m \mu_t$ for $t = 0, 1, \ldots, T$. Since the only effect of μ or p^m on f_t is through w_t, the result follows immediately. Q.E.D.

2.4. Definition. The tax-transfer policy $\mu \in \mathfrak{M}$ is said to be *bonafide* [for the economy defined by the demand functions $f = (f_0, \ldots, f_t, \ldots, f_T)$ and endowments $\omega \in X$] if there is a *proper* monetary equilibrium associated with (ω, μ), that is, there is a $q = (p, p^m) \in \mathcal{Q}(\omega, \mu)$ such that $p^m \neq 0$.

2.5. Proposition. The set of bonafide tax-transfer policies is a cone in $\mathfrak{M}$.

Proof: The proof follows directly from Definition 2.4 and Proposition 2.3. Q.E.D.

Following Balasko and Shell (1981), we go on to study a "cross section" of the cone of bonafide tax-transfer policies. This is the purpose of the next definition.

2.6. Definition. The *bonafide tax-transfer policy* $\mu \in \mathfrak{M}$ is said to be *normalized* if $(p, 1)$ is a (proper) monetary equilibrium associated with (ω, μ). Let $\mathfrak{M}_B(\omega) \subset \mathfrak{M}$ denote the *set of normalized bonafide tax-transfer policies*. [The set of bonafide tax-transfer policies is the positive cone in $\mathfrak{M}$ generated by $\mathfrak{M}_B(\omega)$.]

We next define the set of equilibrium money prices for the economy defined by (ω, μ) and proceed to establish its close relationship with the set $\mathfrak{M}_B(\omega)$.

2.7. Definition. The set of equilibrium money prices $\mathcal{P}^m(\omega, \mu) \subset \mathbb{R}_+$ is defined by

$$\mathcal{P}^m(\omega, \mu) = \{p^m \mid (p, p^m) \in \mathcal{Q}(\omega, \mu) \text{ for some } p \in \mathcal{P}\}.$$

2.8. Proposition. (i) The set $\mathcal{P}'''(\omega,\mu)$ is not empty. (ii) $\mathcal{P}'''(\omega,\mu)\neq\{0\}$ if and only if μ is a bonafide tax-transfer policy. (iii) Fix $\mu\in\mathfrak{M}$ and define $L(\mu)\subset\mathfrak{M}$, the nonnegative ray generated by μ, by $L(\mu)=\{\lambda\mu\mid\lambda\in\mathbb{R}_+\}$. The set $\mathcal{P}'''(\omega,\mu)$ is related to the set $(\mathfrak{M}_B(\omega)\cap L(\mu))$ by a one-to-one mapping.[2]

Proof: (i) Consider the economy without taxes and transfers, $\mu=0$. (a) First, consider the finite case, $1\leq T<+\infty$. Consumers are indirectly resource related in the no-taxation economy. Clearly, an ordinary competitive equilibrium price vector $p\in\mathcal{P}$ exists. (b) Second, consider the infinite case, $T=+\infty$. From Proposition 3.10 (Balasko and Shell 1980, p. 289), we know that a competitive equilibrium price sequence $p\in\mathcal{P}$ exists for the no-taxation economy. Therefore, for T finite or infinite, we know there is a $p\in\mathcal{P}$ such that $(p,1)\in\mathcal{Q}(\omega,0)$ and $(p,0)\in\mathcal{Q}(\omega,\mu)$ (using Proposition 2.3). Hence, we have $0\in\mathcal{P}'''(\omega,\mu)$ (from Definition 2.7). (ii) The proof of (ii) follows directly from Definitions 2.4 and 2.7 and Proposition 2.8.i. (iii) From Definitions 2.4 and 2.7 and Proposition 2.3, the mapping defined by $p'''\mapsto p'''\mu$ is clearly a bijection from $\mathcal{P}'''(\omega,\mu)$ to $\mathfrak{M}_B(\omega)\cap L(\mu)$. Q.E.D.

3 The finite-horizon, overlapping-generations model

Throughout this section, T is assumed to be a positive finite integer. There are then $T+1$ consumers, $h=0,1,\ldots,T$, and $T+1$ periods, $t=1,\ldots,T+1$. Perfect foresight is assumed. Balanced fiscal policies play a central role in finite economies; the formal definition is given next.

3.1. Definition. Let T be finite. The tax-transfer policy

$$\mu=(\mu_0,\ldots,\mu_t,\ldots,\mu_T)\in\mathbb{R}^{T+1}$$

is said to be *balanced* if we have $\sum_{t=0}^{T}\mu_t=0$. The set of balanced tax-transfer policies is $\{\mu\in\mathbb{R}^{T+1}\mid\sum_{t=0}^{T}\mu_t=0\}$.

The amount of outstanding government debt at the end of the last period is equal to $\sum_{t=0}^{T}\mu_t$. Hence, there is an equivalent definition of balancedness applicable to finite-horizon dynamic economies: *The government's tax-transfer policy is said to be balanced if the outstanding debt of the government is zero on the terminal date.*

The next proposition provides the first step in our program of analyzing the relationships between bonafide and balanced tax-transfer policies.

3.2. Proposition. Let T be finite. If the tax-transfer policy $\mu \in \mathbb{R}^{T+1}$ is bonafide (Definition 2.4), then μ is balanced (Definition 3.1).

Proof: Let $q=(p, p^m)$ be a competitive equilibrium for the economy defined by (ω, μ); hence we have $q \in \mathcal{Q}(\omega, \mu)$. Since utility functions are increasing, it follows that $p \cdot f_t(p, w_t) = w_t = p \cdot \omega_t + p^m \mu_t$ for $t = 0, 1, \ldots, T$. Summing these equalities yields

$$p \cdot \sum_{t=0}^{T} f_t(p, w_t) = p \cdot \sum_{t=0}^{T} \omega_t + p^m \sum_{t=0}^{T} \mu_t .$$

But we know from Definition 2.1 that

$$\sum_{t=0}^{T} f_t(p, w_t) = \sum_{t=0}^{T} \omega_t .$$

Therefore, we have $p^m \sum_{t=0}^{T} \mu_t = 0$. Thus, if μ is bonafide [i.e., if there is a proper competitive equilibrium $q=(p, p^m)$], then μ is balanced.

Q.E.D.

For finite economies, Proposition 3.2 is an immediate consequence of Walras' law. The converse of Proposition 3.2 is also true, but its proof is not so straightforward. In what follows, we show that each balanced policy is bonafide. Our strategy of proof is based on that of Balasko and Shell (1983, Proposition 3.6). We construct an associated no-taxation economy in which endowments of the first commodity have been reallocated. For the static economy (Balasko and Shell 1983), suffiiciently small reallocations can be made while retaining the property that individual endowments lie within the consumption set. This is not in general possible in the dynamic economy because of the nature of generational overlap. Hence, in the analysis of the associated economy, we must expand the consumption sets and extend the preferences to be defined over the expanded consumption sets.

We proceed to the construction of the associated no-taxation economy. Let $\mu = (\mu_0, \mu_1, \ldots, \mu_t, \ldots, \mu_T)$ be a vector in $\mathbb{R}^{T+1}$ that satisfies $\sum_{t=0}^{T} \mu_t = 0$, where T is finite. Let $\bar{\omega} = (\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_t, \ldots, \bar{\omega}_T)$ be the vector of endowments for the associated economy that satisfies

$$(3.3) \qquad \begin{aligned} \bar{\omega}_0 &= (\omega_0^{1,1} + \mu_0, \omega_0^{1,2}, \ldots, \omega_0^{1,\ell}, 0, \ldots, 0), \\ \bar{\omega}_1 &= (\omega_1^{1,1} + \mu_1, \omega_1^{1,2}, \ldots, \omega_0^{1,\ell}, \omega_1^{2,1}, \ldots, \omega_1^{2,\ell}, 0, \ldots, 0), \\ &\vdots \\ \bar{\omega}_t &= (\mu_t, 0, \ldots, 0, \omega_t^{t,1}, \ldots, \omega_t^{t,\ell}, \omega_t^{t+1,1}, \ldots, \omega_t^{t+1,\ell}, 0, \ldots, 0), \\ &\vdots \\ \bar{\omega}_T &= (\mu_T, 0, \ldots, 0, \omega_T^{T,1}, \ldots, \omega_T^{T,\ell}, \omega_T^{T+1,1}, \ldots, \omega_T^{T+1,\ell}), \end{aligned}$$

where the omegas on the right side are the endowments

$$\omega = (\omega_0, \omega_1, \dots, \omega_t, \dots, \omega_T) \in X$$

for the original (tax-transfer) economy described in Section 2. Notice that by construction we have $\tilde{\omega}_t \in \mathbb{R}^{\ell(T+1)}$ for $t = 0, 1, \dots, T$, and

$$\sum_{t=0}^{T} \tilde{\omega}_t = \sum_{t=0}^{T} \omega_t = r \in \mathbb{R}^{\ell(T+1)}_{++},$$

even though some of the $\tilde{\omega}_t$ may have nonpositive first components.

In Section 2, the preferences of consumer $t = 1, \dots, T$ are defined only on $\mathbb{R}^{2\ell}_{++}$ and those of consumer 0 only on $\mathbb{R}^{\ell}_{++}$. Here we extend preferences so that they are defined on all of $\mathbb{R}^{\ell(T+1)}$. This is done in two steps. In the first step, preferences are extended from the positive orthant, $\mathbb{R}^{2\ell}_{++}$ or $\mathbb{R}^{\ell}_{++}$, to the whole Euclidean space, $\mathbb{R}^{2\ell}$ or $\mathbb{R}^{\ell}$. In the second step, preferences are extended from $\mathbb{R}^{2\ell}$ or $\mathbb{R}^{\ell}$ to $\mathbb{R}^{\ell(T+1)}$.

Step 1: Consider consumer t ($t = 1, \dots, T$). Through each point (x_t^t, x_t^{t+1}) in $\mathbb{R}^{2\ell}_{++}$ passes an indifference surface that is a level set to $u_t(\cdot)$. First, extend the preference map to $\mathbb{R}^{2\ell}_{+}$. The set $\mathbb{R}^{2\ell}_{+} \setminus \mathbb{R}^{2\ell}_{++}$ is assumed to be an indifference surface. Each point in $\mathbb{R}^{2\ell}_{++}$ is preferred to any point in $\mathbb{R}^{2\ell}_{+} \setminus \mathbb{R}^{2\ell}_{++}$. Then construct a family of piecewise linear indifference surfaces in $\mathbb{R}^{2\ell} \setminus \mathbb{R}^{2\ell}_{+}$, where each indifference surface in $\mathbb{R}^{2\ell} \setminus \mathbb{R}^{2\ell}_{+}$ is parallel to the indifference surface $\mathbb{R}^{2\ell}_{+} \setminus \mathbb{R}^{2\ell}_{++}$. Preferences on $\mathbb{R}^{2\ell}$ are thus monotonic (although not strictly monotonic) and exhibit local nonsatiation. Also, preferred sets in $\mathbb{R}^{2\ell}$ are convex. [The logic of this extension is illustrated in Figure 1. The solid curves depict indifference surfaces generated by the utility function $u_t(\cdot)$. The dashed curves represent the new indifference surfaces, which expand the preference relation to all of $\mathbb{R}^{2\ell}$.] In like fashion, extend the preferences of consumer 0 from $\mathbb{R}^{\ell}_{++}$ to all of $\mathbb{R}^{\ell}$.

The extended preferences can be represented by continuous utility functions $\phi_0 : \mathbb{R}^{\ell} \to \mathbb{R}$ and $\phi_t : \mathbb{R}^{2\ell} \to \mathbb{R}$ for $t = 1, \dots, T$, where

(3.4) $$\phi_0(x_0) = u_0(x_0) \quad \text{for } x_0 \in \mathbb{R}^{\ell}_{++},$$

$$\phi_0(x_0) = \min_{i=1,\dots,\ell} (x_0^{1,i}) \quad \text{for } x_0 \in \mathbb{R}^{\ell} \setminus \mathbb{R}^{\ell}_{++},$$

and

$$\phi_t(x_t) = u_t(x_t) \quad \text{for } x_t \in \mathbb{R}^{2\ell}_{++},$$

$$\phi_t(x_t) = \min_{\substack{s=t,t+1 \\ i=1,\dots,\ell}} (x_t^{s,i}) \quad \text{for } x_t \in \mathbb{R}^{2\ell} \setminus \mathbb{R}^{2\ell}_{++},$$

for $t = 1, \dots, T$.

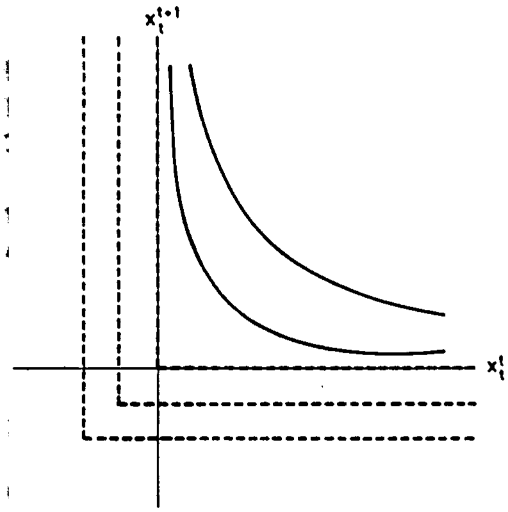

Figure 1. Extension of preferences from $\mathbb{R}^{2\ell}_{++}$ to $\mathbb{R}^{2\ell}$.

For purposes of the construction in equations (3.4), the functions u_t have been scaled so that

(i) $u_0(x_0) \in \mathbb{R}_{++}$ for each $x_0 \in \mathbb{R}^{\ell}_{++}$, and $u_t(x_t) \in \mathbb{R}_{++}$ for each $x_t \in \mathbb{R}^{2\ell}_{++}$ $(t = 1, \dots, T)$,

and

(ii) if $\{x_0^\nu\}$ (respectively, $\{x_t^\nu\}$) is a sequence with $x_0^\nu \in \mathbb{R}^{\ell}_{++}$ (respectively, $x_t^\nu \in \mathbb{R}^{2\ell}_{++}$) for each ν and $\lim_{\nu \to \infty} x_0^\nu \in \mathbb{R}^{\ell}_{+} \backslash \mathbb{R}^{\ell}_{++}$ (respectively, $\lim_{\nu \to \infty} x_t^\nu \in \mathbb{R}^{2\ell}_{+} \backslash \mathbb{R}^{2\ell}_{++}$), then $\lim_{\nu \to \infty} u_0(x_0^\nu) = 0$ (respectively, $\lim_{\nu \to \infty} u_t(x_t^\nu) = 0$ for $t = 1, \dots, T$).

Step 2: Let $\tilde{x}_t = (x_t^1, \dots, x_t^t, x_t^{t+1}, \dots, x_t^{T+1}) \in \mathbb{R}^{\ell(T+1)}$ be consumer t's extended consumption vector. Let his preferences on $\mathbb{R}^{\ell(T+1)}$ be represented be the continuous utility function $v_t : \mathbb{R}^{\ell(T+1)} \to \mathbb{R}$, which is defined by

$$v_t(\tilde{x}_t) = \phi_t(x_t) - b \sum_{\substack{s \neq t, s \neq t+1 \\ i = 1, \dots, \ell}} |x_t^{i,s}| \quad \text{for } t = 1, \dots, T \tag{3.5}$$

and

$$v_0(\tilde{x}_0) = \phi_0(x_0) - b \sum_{\substack{s = 2, \dots, T \\ i = 1, \dots, \ell}} |x_0^{i,s}|,$$

where b is a positive scalar and $|\ |$ denotes absolute value. The utility function v_t is nondecreasing in each of the commodities available during consumer t's lifetime and is strictly decreasing in the absolute value of consumptions outside his lifetime. Preferences defined on $\mathbb{R}^{\ell(T+1)}$ are continuous, convex, and exhibit local nonsatiation. Notice also that the preorderings defined by equations (3.4) on $\mathbb{R}^{2\ell}$ or $\mathbb{R}^{\ell}$ are preserved on $\mathbb{R}^{\ell(T+1)}$ [cf. (3.5)]; in particular, we have

(3.6) $(0,\dots,0,x_t^t,x_t^{t+1},0,\dots,0)$ is preferred to (respectively, indifferent to) $(0,\dots,0,y_t^t,y_t^{t+1},0,\dots,0)$ on $\mathbb{R}^{\ell(T+1)}$ if and only if (x_t^t,x_t^{t+1}) is preferred to (respectively, indifferent to) (y_t^t,y_t^{t+1}) on $\mathbb{R}^{2\ell}$ for consumer $t=1,\dots,T$,

and

$(x_0^1,0,\dots,0)$ is preferred to (respectively, indifferent to) $(y_0^1,0,\dots,0)$ on $\mathbb{R}^{\ell(T+1)}$ if and only if x_0^1 is preferred to (respectively, indifferent to) y_0^1 on $\mathbb{R}^{\ell}$ for consumer 0.

Other things equal, consumer t prefers zero consumption outside his lifetime to nonzero consumption outside his lifetime; that is, we have, from equation (3.5),

(3.7) $(0,\dots,0,x_t^t,x_t^{t+1},0,\dots,0)\succsim_t(x_t^1,\dots,x_t^t,x_t^{t+1},\dots,x_t^{T+1})$ with strict preference when the two vectors are unequal for consumer $t=1,\dots,T$,

and

$(x_0^1,0,\dots,0)\succsim_0(x_0^1,\dots,x_0^{T+1})$ with strict preference when the two vectors are unequal for consumer 0.

Because of property 3.7, we know that consumers will not choose positive consumptions outside their natural lifetimes as long as prices are nonnegative. They might, however, choose some negative consumptions outside their lifetimes in order to finance increased consumption during their natural life spans.

This bizarre behavior will eventually be ruled out by specifying a sufficiently large value for the scalar b. Until this is accomplished, we must restrict consumer t's consumption to K_t, a compact subset of $\mathbb{R}^{\ell(T+1)}$, defined as follows: Given a positive scalar ϵ, define the negative scalar a by

(3.8) $$a=\min_{t=0,\dots,T}(\mu_t)-\epsilon.$$

It follows from equation (3.3) that $\tilde{\omega}_t^{i,s} > a$ for $i = 1, \ldots, \ell$; $s = 1, \ldots, T+1$; and $t = 0, \ldots, T$. Next define the positive scalar c by

$$(3.9) \qquad c = \max_{\substack{i=1,\ldots,\ell \\ s=1,\ldots,T+1}} (r^{i,s}) + \epsilon - Ta.$$

Then define the set $K_t \subset \mathbb{R}^{\ell(T+1)}$ by

$$(3.10) \qquad K_t = \{\tilde{x}_t \in \mathbb{R}^{\ell(T+1)} \mid a \leq x_t^{i,s} \leq c \text{ for } i = 1, \ldots, \ell \text{ and } s = 1, \ldots, T+1\},$$

for $t = 0, \ldots, T$. The set K_t is obviously compact (and independent of t). We restrict $\tilde{x}_t$ to K_t. This restriction is essential in the associated no-taxation economy because without it there could be the possibility of negative consumptions that are unbounded.

Let $p = (p^1, \ldots, p^{T+1})$ be the commodity price vector. Restrict prices in the associated economy to the normalized set

$$\tilde{\mathcal{P}} = \left\{ p \in \mathbb{R}_+^{\ell(T+1)} \,\middle|\, \sum_{t,i} p^{t,i} = 1 \right\}.$$

Let $w_t = p \cdot \tilde{\omega}_t \in \mathbb{R}$ be the wealth of consumer t, $t = 0, 1, \ldots, T$. Consumer t is assumed to maximize $v_t(\tilde{x}_t)$ subject to the constraints $\tilde{x}_t \in K_t$ and $p \cdot \tilde{x}_t \leq p \cdot \tilde{\omega}_t = w_t$. This consumer's demands can then be described by the upper hemi-continuous correspondence $\tilde{f}_t : (p, w_t) \mapsto \tilde{x}_t$ so that $\tilde{f}_t : \tilde{\mathcal{P}} \times \mathbb{R} \to K_t \subset \mathbb{R}^{\ell(T+1)}$.

We next define competitive equilibrium for the associated no-taxation economy.

3.11. Definition. Competitive equilibrium for the associated (finite) economy with endowments $\tilde{\omega} \in \mathbb{R}^{\ell(T+1)(T+1)}$ is a price vector $p \in \tilde{\mathcal{P}}$ satisfying

$$\sum_{t=0}^{T} \tilde{f}_t(p, p \cdot \tilde{\omega}_t) \leq \sum_{t=0}^{T} \tilde{\omega}_t$$

with strict equality for component (t, i) if $p^{t,i} > 0$, $t = 1, \ldots, T+1$ and $i = 1, \ldots, \ell$.

3.12. Lemma. Consider the associated economy with endowments $\tilde{\omega} \in \mathbb{R}^{\ell(T+1)(T+1)}$ satisfying the system (3.3). There is a price vector $p \in \tilde{\mathcal{P}}$ that is a competitive equilibrium for the associated economy (Definition 3.11).

Proof: First, we use the theorem in Debreu (1962, pp. 259–260) to establish the existence of a compensated equilibrium (or quasiequilibrium), $p \in \tilde{\mathcal{P}}$.

(i) The set K_t is closed, convex, and bounded by construction (3.8)-(3.10). (ii) Because of (3.5) and (3.8)-(3.10), we know that for each attainable $\tilde{x}_t$ in K_t there is a $\tilde{y}_t$ in K_t that is preferred by consumer t to $\tilde{x}_t$. (iii) Also, for every $\tilde{y}_t$ in K_t, the set $\{\tilde{x}_t \in K_t \mid \tilde{x}_t \succsim_t \tilde{y}_t\}$ is closed in K_t and is convex. We have thus established the existence of a compensated equilibrium.

Second, observe that each individual endowment vector lies in the interior of the consumption set. In particular, $\tilde{\omega}_t$ is in the interior of K_t for $t = 0, 1, \ldots, T$. Thus, consumers are (directly) resource related to one another (cf. Arrow and Hahn 1971, Chapter 5). Thus the compensated equilibrium $p \in \tilde{\mathcal{P}}$ is a competitive equilibrium (Definition 3.11).

Q.E.D.

In equilibrium, some (but not all) of the components of the price vector $p \in \tilde{\mathcal{P}}$ may be zero. The corresponding consumption allocation vector $\tilde{x} = (\tilde{x}_0, \ldots, \tilde{x}_T) \in (\tilde{f}_0(p, p \cdot \tilde{\omega}_0), \ldots, \tilde{f}_T(p, p \cdot \tilde{\omega}_T))$ contains components that are nonpositive. (In particular, we have as a consequence of the nonnegativity of prices, Property 3.7 and Definition 3.11 that $x_t^s \leq 0$ for $t = 1, \ldots, T$; $s \neq t$, $s \neq t+1$, and $x_0^s \leq 0$ for $s \neq 1$.) The corresponding vector of individual wealths $w = (w_0, \ldots, w_T)$ may contain some zero components and some negative components. We next establish that the first component of p is necessarily positive.

3.13. Lemma. Let p be a competitive equilibrium for the associated economy (Definition 3.11). Then the first component of p is positive; that is, $p^{1,1} \in \mathbb{R}_{++}$.

Proof: Assume otherwise; that is, assume that $p^{1,1} = 0$. This implies that $p^1 = 0$ and $p^2 = 0$; otherwise, w_1 would be positive, which implies that $x_1 = (x_1^1, x_1^2)$ would in equilibrium be a strictly positive vector. Then (p^1, p^2) would be proportional to grad $u_1(x_1^1, x_1^2)$. Because of the assumptions made on u_1, if any component of (p^1, p^2) is positive, all components must be positive.

Assume that $p^t = 0$ for some t $(t = 2, \ldots, T)$. Under the assumption that $p^{1,1} = 0$, we have that unless (p^t, p^{t+1}) is identically zero, then w_t is positive and thus (x_t^t, x_t^{t+1}) is strictly positive. Hence, (p^t, p^{t+1}) must be proportional to grad $u_t(x_t^t, x_t^{t+1})$. Therefore, $p^t = 0$ implies $p^{t+1} = 0$ for $t = 2, \ldots, T$.

We have established that $p^{1,1} = 0$ implies $p = 0$. The zero vector does not belong to $\tilde{\mathcal{P}}$ because $0 + \cdots + 0 \neq 1$. We have a contradiction. Hence, we have established that $p^{1,1}$ is positive. Q.E.D.

Nonpositive wealths and zero prices must be allowed for in the associated economy because negative endowments of commodity (1, 1) are permitted. Hence, in equilibrium, nonpositive consumptions during consumers' natural life spans are possible. If, however, the balanced tax-transfer vector $\mu=(\mu_0,\ldots,\mu_T)$ were sufficiently close to zero, then one would expect equilibrium prices to be strictly positive, equilibrium wealths to be strictly positive, and hence equilibrium consumptions during consumers' lifetimes to be strictly positive. Having found a set of balanced tax-transfer vectors sufficiently close to zero, one could then choose a sufficiently large value for the scalar b so that in equilibrium consumption of goods outside the consumers' lifetimes is zero (and so that in equilibrium the restrictions to the sets K_t are inessential). These ideas are formalized in the next proposition, which can be taken as an extension of Proposition 3.6 in Balasko and Shell (1983).

Define the sequence $\{\mu^\nu\}=\{(\mu_0^\nu,\ldots,\mu_T^\nu)\}$ where $\mu^\nu\in\mathbb{R}^{T+1}$ and $\sum_{s=0}^T\mu_s^\nu=0$. Assume that μ^ν converges pointwise to 0 as $\nu\to\infty$. Let $p^\nu\in\tilde{\mathcal{P}}$ be the equilibrium price vector for the associated economy (Definition 3.11) when $(\mu_0,\ldots,\mu_T)=(\mu_0^\nu,\ldots,\mu_T^\nu)$ and ω is fixed. Define in the natural way the corresponding endowment vector $\tilde{\omega}^\nu=(\tilde{\omega}_0^\nu,\ldots,\tilde{\omega}_T^\nu)\in\mathbb{R}^{\ell(T+1)(T+1)}$, the corresponding equilibrium consumption allocations $\tilde{x}^\nu=(\tilde{x}_0^\nu,\ldots,\tilde{x}_T^\nu)\in\mathbb{R}^{\ell(T+1)(T+1)}$ and $x^\nu=(x_0^\nu,\ldots,x_T^\nu)\in\mathbb{R}^{\ell(2T+1)}$ and so forth.

3.14. Proposition. Consider the sequence of vectors $\{\mu^\nu\}=\{(\mu_0^\nu,\ldots,\mu_T^\nu)\}$ satisfying $\sum_{t=0}^T\mu_t^\nu=0$ for each $\nu=1,2,\ldots$ and that converges pointwise to zero as $\nu\to+\infty$. Then there is a ν^* such that for $\nu\geq\nu^*$, the corresponding equilibrium price vector p^ν and consumption allocation vector x^ν in the associated economy are each strictly positive. Thus, we have $p^\nu\in\mathbb{R}_{++}^{\ell(T+1)}$ and $x^\nu\in\mathbb{R}_{++}^{\ell(2T+1)}$ for $\nu\geq\nu^*$. Furthermore, if the scalar b [defined in property (3.5)] is sufficiently large, then the corresponding equilibrium extended allocation vector $\tilde{x}^\nu$ is zero on all but $\ell(2T+1)$ components for $\nu\geq\nu^*$.

Proof: As $\nu\to+\infty$, $\omega_0^{1,1}+\mu_0^\nu\to\omega_0^{1,1}$. Hence, there is a ν_0 such that for $\nu\geq\nu_0$, $(x_0^1)^\nu$ belongs to X_0, a nonempty, compact subset of $\mathbb{R}_{++}^\ell$ defined by

$$X_0=\{x_0^1\,|\,x_0^{1,i}\leq c\ \text{ for } i=1,\ldots,\ell \text{ and } u_0(x_0^1)\geq u_0(\omega_0^1/2)\},$$

where $c\in\mathbb{R}_{++}$ is defined in equation (3.9). For $\nu\geq\nu_0$, the price vector $(p^1)^\nu$ is proportional to grad $u_0((x_0^1)^\nu)\in\mathbb{R}_{++}^\ell$. By Lemma 3.12, $(p^{1,1})^\nu\in\mathbb{R}_{++}$ for each ν. Hence, we have $(p^1)^\nu\in\mathbb{R}_{++}^\ell$ for $\nu\geq\nu_0$. Furthermore, since u_0 is strictly quasiconcave, we know that there are vectors $\alpha^1=$

$(\alpha^{1,1},\dots,\alpha^{1,\ell})\in\mathbb{R}^{\ell}_{++}$ and $\beta^1=(\beta^{1,1},\dots,\beta^{1,\ell})\in\mathbb{R}^{\ell}_{++}$ with the property

$$0<\alpha^1\le(p^1)^\nu/(p^{1,1})^\nu\le\beta^1<+\infty$$

for $\nu\ge\nu_0$ [cf. Arrow and Hahn (1971, pp. 29-30, 101-104) and Balasko and Shell (1980, Lemma 3.4, pp. 287-288)].

As $\nu\to+\infty$, $\omega_1^{1,1}+\mu_1^\nu\to\omega_1^{1,1}$. Hence, there is $\nu_1\ge\nu_0$ such that for $\nu\ge\nu_1$, $((x_1^1)^\nu,(x_1^2)^\nu)$ belongs to X_1, a nonempty, compact subset of $\mathbb{R}^{2\ell}_{++}$ defined by

$$X_1=\{(x_1^1,x_1^2)\mid x_1^{s,i}\le c \text{ for } s=1,2;\ i=1,\dots,\ell, \text{ and}$$
$$u_1(x_1^1,x_1^2)\ge u_1(\omega_1^1/2,\omega_1^2)\},$$

where $c\in\mathbb{R}_{++}$ is defined in equation (3.9). For $\nu\ge\nu_1$, the price vector $((p^1)^\nu,(p^2)^\nu)$ is proportional to grad $u_1((x_1^1)^\nu,(x_1^2)^\nu)\in\mathbb{R}^{2\ell}_{++}$. Since $(p^1)^\nu\in\mathbb{R}^{\ell}_{++}$, we have that $(p^2)^\nu\in\mathbb{R}^{\ell}_{++}$ for $\nu\ge\nu_1$. Furthermore, there are vectors $\alpha^2=(\alpha^{2,1},\dots,\alpha^{2,\ell})\in\mathbb{R}^{\ell}_{++}$ and $\beta^2=(\beta^{2,1},\dots,\beta^{2,\ell})\in\mathbb{R}^{\ell}_{++}$ with the property

$$0<\alpha^2\le(p^2)^\nu/(p^{1,1})^\nu\le\beta^2<+\infty$$

for $\nu\ge\nu_1$.

Assume that for some particular t $(t=2,\dots,T)$ there is ν_{t-1} such that for $\nu\ge\nu_{t-1}$ we have

$$0<\alpha^t\le(p^t)^\nu/(p^{1,1})^\nu\le\beta^t<+\infty,$$

where $\alpha^t=(\alpha^{t,1},\dots,\alpha^{t,\ell})\in\mathbb{R}^{\ell}_{++}$ and $\beta^t=(\beta^{t,1},\dots,\beta^{t,\ell})\in\mathbb{R}^{\ell}_{++}$. Wealth of consumer t is given by

$$w_t^\nu=(p^\nu)\cdot(\tilde{\omega}_t^\nu)=(p^{1,1})^\nu\mu_t^\nu+(p^t)^\nu\cdot\omega_t^t+(p^{t+1})^\nu\cdot\omega_t^{t+1}$$
$$\ge(p^t)^\nu\cdot(\omega_t^{t,1}+\min(0,\mu_t^\nu/\alpha^{t,1}),\omega_t^{t,2},\dots,\omega_t^{t,\ell})+(p^{t+1})^\nu\cdot\omega_t^{t+1}$$

for $\nu\ge\nu_{t-1}$. Hence there is a $\nu_t\ge\nu_{t-1}$ such that $((x_t^t)^\nu,(x_t^{t+1})^\nu)$ belongs to X_t, a nonempty, compact subset of $\mathbb{R}^{2\ell}_{++}$ defined by

$$X_t=\{(x_t^t,x_t^{t+1})\mid x_t^{s,i}\le c \text{ for } s=t,t+1,\ i=1,\dots,\ell, \text{ and}$$
$$u_t(x_t^t,x_t^{t+1})\ge u_t(\omega_t^t/2,\omega_t^{t+1})\}$$

for $\nu\ge\nu_t$, where $c\in\mathbb{R}_{++}$ is defined in equation (3.9). It follows that $(p^{t+1})^\nu\in\mathbb{R}^{\ell}_{++}$ for $\nu\ge\nu_t$ and furthermore that there are vectors $\alpha^{t+1}=(\alpha^{t+1,1},\dots,\alpha^{t+1,\ell})\in\mathbb{R}^{\ell}_{++}$ and $\beta^{t+1}=(\beta^{t+1,1},\dots,\beta^{t+1,\ell})\in\mathbb{R}^{\ell}_{++}$ with the property

$$0<\alpha^{t+1}\le(p^{t+1})^\nu/(p^{1,1})^\nu\le\beta^{t+1}<+\infty$$

for $\nu\ge\nu_t$.

Let $\nu^* = \nu_T$. The proof by induction is complete. We have established that there is a ν^* with the property that $x^\nu \in \mathbb{R}_{++}^{\ell(2T+1)}$, $p^\nu \in \mathbb{R}_{++}^{\ell(T+1)}$, and $w^\nu \in \mathbb{R}_{++}^{T+1}$ for $\nu \geq \nu^*$.

Since grad u_t is a continuous function of x_t on the compact set X_t, we can provide an upper bound on marginal utilities. In particular, we know there is a positive scalar γ with the properties

$$\text{(i)} \qquad \gamma > \frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{s,i}} \quad \text{for } s=t,\ s=t+1;\ i=1,\dots,\ell;$$
$$(x_t^t, x_t^{t+1}) \in X_t;\ t=1,\dots,T$$

and

$$\text{(ii)} \qquad \gamma > \frac{\partial u_0(x_0^1)}{\partial x_0^{1,i}} \quad \text{for } i=1,\dots,\ell;\ x_0^1 \in X_0.$$

Let $\alpha^{t,i}$ and $\beta^{t,i}$ be the upper and lower bounds on the price ratio $p^{t,i}/p^{1,1}$ that were derived in the first part of the proof. Choose positive scalars α and β that satisfy $\alpha < \alpha^{t,i}$ and $\beta > \beta^{t,i}$ for $i=1,\dots,\ell$; $t=1,\dots,T+1$. Then define the scalar b by

$$(3.15) \qquad b = \gamma\beta/\alpha.$$

Let the extended utility functions v_t defined in equations (3.5) satisfy the restriction (3.15). Then for $\nu \geq \nu^*$, equilibrium consumptions outside natural life spans are zero. Hence, in equilibrium, the extended allocation vector $\tilde{x}^\nu$ is zero on $\ell(T+1)(T+1) - \ell(2T+1)$ components and is strictly positive on the remaining $\ell(2T+1)$ components for $\nu \geq \nu^*$. Q.E.D.

In the next lemma, we formalize the relationship between competitive equilibrium in the original economy (with taxation) and competitive equilibrium in the associated (no-taxation) economy.

3.16. Lemma. Let T be finite.

(i) Consider the associated no-taxation economy defined by the extended endowment vector $\tilde{\omega} \in \mathbb{R}^{\ell(T+1)(T+1)}$, which is consistent with the construction in (3.3). Assume that there is a competitive equilibrium price vector $p \in \tilde{\mathcal{P}}$ (Definition 3.11). Assume further that p is strictly positive, that is, $p \in \tilde{\mathcal{P}} \cap \mathbb{R}_{++}^{\ell(T+1)}$, equilibrium wealths are strictly positive, that is, $w = (p\cdot\tilde{\omega}_0, \dots, p\cdot\tilde{\omega}_t, \dots, p\cdot\tilde{\omega}_T) \in \mathbb{R}_{++}^{T+1}$, and the equilibrium allocation vector x is strictly positive, that is, $x = (x_0, \dots, x_t, \dots, x_T) \in \mathbb{R}_{++}^{\ell(2T+1)}$, but that the extended equilibrium allocation vector $\tilde{x} = (\tilde{x}_0, \dots, \tilde{x}_t, \dots, \tilde{x}_T) \in \mathbb{R}^{\ell(T+1)(T+1)}$ is zero on all but $\ell(2T+1)$ components. Define the price vector $p' \in \mathcal{P}$ by $p' = p/p^{1,1}$. Then $q = (p', 1) \in \mathcal{Q}$ is a competitive equilibrium for the taxation economy (Definition 2.1), where the endowments $\omega \in \mathbb{R}_{++}^{\ell(T+1)}$ together with the balanced tax-transfer policy $\mu \in \mathbb{R}^{T+1}$ satisfy equations (3.3); thus, we have $q = (p', 1) \in \mathcal{Q}(\omega, \mu)$.

(ii) Consider the taxation economy defined by the endowment vector $\omega \in \mathbb{R}_{++}^{\ell(2T+1)}$ and the balanced tax-transfer policy $\mu \in \mathbb{R}^{T+1}$. Assume that there is an equilibrium in which the price of money is unity; that is, assume there is $p \in \mathcal{P}$ with the property $(p, 1) \in \mathcal{Q}(\omega, \mu)$ (Definition 2.1). Define the price vector p' by $p' = p/\sum_{t,i} p^{t,i}$ so that $p' \in \bar{\mathcal{P}} \cap \mathbb{R}_{++}^{\ell(T+1)}$. Then p' is a competitive equilibrium for the associated economy (Definition 3.11), where the extended endowment vector $\tilde{\omega} \in \mathbb{R}^{\ell(T+1)(T+1)}$ is related to the balanced tax-transfer vector $\mu \in \mathbb{R}^{T+1}$ by equation (3.3), and the scalar b [equation (3.5)] is sufficiently large.

Proof: (i) Fix $\tilde{\omega}$, the extended endowment vector for the associated economy. Choose $\omega \in \mathbb{R}_{++}^{\ell(2T+1)}$ and $\mu = (\mu_0, \dots, \mu_t, \dots, \mu_T) \in \mathbb{R}^{T+1}$, which together satisfy equations (3.3) and $\sum_{t=0}^{T} \mu_t = 0$. Let $p^m = 1$, and define the vector of wealths

$$
\begin{aligned}
w' &= (p' \cdot \omega_0 + \mu_0, \dots, p' \cdot \omega_t + \mu_t, \dots, p' \cdot \omega_t + \mu_t) \\
&= (p' \cdot \tilde{\omega}_0, \dots, p' \cdot \tilde{\omega}_t, \dots, p' \cdot \tilde{\omega}_T) \\
&= (1/p^{1,1})(p \cdot \tilde{\omega}_0, \dots, p \cdot \tilde{\omega}_t, \dots, p \cdot \tilde{\omega}_T) = (1/p^{1,1}) w.
\end{aligned}
$$

Since (p', w') is proportional to (p, w), it follows that $f_0^{1,i}(p', w'_0) = \bar{f}_0^{1,i}(p, w_0) \in \mathbb{R}_{++}$ for $i = 1, \dots, \ell$ and $f_t^{s,i}(p', w'_t) = \bar{f}_t^{s,i}(p, w_t) \in \mathbb{R}_{++}$ for $s = t, t+1$; $i = 1, \dots, \ell$; $t = 1, \dots, T$. It follows from definitions 2.1 and 3.11 that $q = (p', 1) \in \mathcal{Q}(\omega, \mu)$. (ii) Let

$$x = (x_0, \dots, x_t, \dots, x_T) = (x_0^1, \dots, x_t^t, x_t^{t+1}, \dots, x_T^T, x_T^{T+1}) \in \mathbb{R}_{++}^{\ell(2T+1)}$$

be the competitive allocation in the taxation economy with $p^m = 1$. Define the scalar b [cf. equation (3.5)] as in equation (3.15) except that the sets X_t $(t = 0, \dots, T)$ are replaced by the sets Y_t $(t = 0, \dots, T)$ where

$$Y_0 = \{ y_0^1 \in \mathbb{R}_{++}^{\ell} \mid y_0^{1,i} \le c \text{ for } i = 1, \dots, \ell \text{ and } u_0(y_0^1) \ge u_0(x_0^1/2) \}$$

and

$$
\begin{aligned}
Y_t = \{ (y_t^t, y_t^{t+1}) \in \mathbb{R}_{++}^{2\ell} \mid y_t^{s,i} \le c \text{ for } s = t, t+1;\ i = 1, \dots, \ell \\
\text{and } u_t(y_t^t, y_t^{t+1}) \ge u_t(x_t^t/2, x_t^{t+1}) \}
\end{aligned}
$$

for $t = 1, \dots, T$. Define w', the vector of wealths in the associated economy, by

$$\begin{aligned} w' = p'\cdot\bar{\omega} &= (p'\cdot\bar{\omega}_0, \ldots, p'\cdot\bar{\omega}_t, \ldots, p'\cdot\bar{\omega}_T) \\ &= (1/(p^{1,1})')(p\cdot\bar{\omega}_0, \ldots, p\cdot\bar{\omega}_t, \ldots, p\cdot\bar{\omega}_T) \\ &= (1/(p^{1,1})')(p\cdot\omega_0+\mu_0, \ldots, p\cdot\omega_t+\mu_t, \ldots, p\cdot\omega_T+\mu_T) = w/(p^{1,1})', \end{aligned}$$

where $w \in \mathbb{R}^{T+1}_{++}$ is the vector of wealths in the taxation economy. Because (p', w') is proportional to (p, w) and b is sufficiently large, we have

$$\begin{aligned} &\bar{f}_0^{i,1}(p', w_0') = f_0^{1,i}(p, w_0) \in \mathbb{R}_{++} \quad \text{for } i = 1, \ldots, \ell, \\ &\bar{f}_t^{s,i}(p', w_t') = f_t^{s,i}(p, w_t) \in \mathbb{R}_{++} \quad \text{for } s = t, s = t+1; \\ &\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, \ell;\ t = 1, \ldots, T, \end{aligned}$$

and

$$\bar{f}_t^{s,i}(p', w_t') = 0 \quad \text{otherwise.}$$

Q.E.D.

In the next three propositions, we complete our analysis (for the finite-horizon taxation economy) of the set of normalized bonafide tax-transfer policies, $\mathfrak{M}_B(\omega)$.

3.17. Proposition. Fix resources $r \in \mathbb{R}^{\ell(T+1)}_{++}$. Any Pareto-optimal allocation is a competitive equilibrium allocation associated with the fixed endowment vector $\omega = (\omega_0, \ldots, \omega_t, \ldots, \omega_T) \in \mathbb{R}^{\ell(2T+1)}_{++}$ satisfying $\sum_{t=0}^{T} \omega_t = r$ and some tax-transfer policy $\mu = (\mu_0, \ldots, \mu_t, \ldots, \mu_T) \in \mathfrak{M}_B(\omega)$.

Proof: Let $\mathfrak{P} \subset \mathbb{R}^{\ell(2T+1)}_{++}$ be the set of Pareto-optimal allocations $x = (x_0, \ldots, x_t, \ldots, x_T)$ defined by the given consumer preferences and the given resources r. Let $g: \mathfrak{P} \to \mathcal{P}$ be the mapping that associates the Pareto-optimal allocation with the unique price vector that supports x. Define $\psi(x)$ by $\psi(x) = g(x)\cdot((x_0-\omega_0), \ldots, (x_t-\omega_t), \ldots, (x_T-\omega_T))$. We have $\psi(x) \in \mathbb{R}^{T+1}$. Let $\mu = (\mu_0, \ldots, \mu_t, \ldots, \mu_T) = \psi(x)$. Because $\sum_{t=0}^{T} x_t = r = \sum_{t=0}^{T} \omega_t$, it follows that $\sum_{t=0}^{T} \mu_t = 0$. Observe that if $q = (p, 1) \in \mathcal{Q}(\omega, \mu)$ is a competitive equilibrium price system, then the competitive equilibrium allocation vector is $(f_0(p, p\cdot\omega_0+\mu_0), \ldots, f_t(p, p\cdot\omega_t+\mu_t), \ldots, f_T(p, p\cdot\omega_T+\mu_T))$. Hence, $\mu = \psi(x)$ is the unique normalized tax-transfer policy that decentralizes the Pareto-optimal allocation x. Q.E.D.

Proposition 3.17 is the lump-sum taxation version of the second fundamental theorem of welfare economics. It is given for completeness and because its proof is useful for the proof of the following proposition, which along with Corollary 3.19 completes our analysis for the finite-horizon economy of the set of normalized bonafide tax-transfer policies.

3.18. Proposition. Let T be finite. The set of normalized bonafide tax-transfer policies, $\mathfrak{M}_B(\omega)$, is bounded, arc connected, and 0 is contained in its relative interior.

Proof: The ambient space is

$$\left\{\mu = (\mu_0, \dots, \mu_t, \dots, \mu_T) \in \mathbb{R}^{T+1} \,\middle|\, \sum_{t=0}^{T} \mu_t = 0\right\}.$$

(i) *Interiority of* 0. If $\mu = 0$ and $p''' = 1$, then $w_t = p \cdot \omega_t$ for $t = 0, 1, \dots, T$. Hence the equations in Definition 2.1 reduce to those for a standard exchange economy without taxation (cf., e.g., Arrow and Hahn 1971, Chapter 5). Thus, there is $p \in \mathbb{R}_{++}^{\ell(T+1)}$ such that $(p, 1) \in \mathcal{Q}(\omega, 0)$. From Proposition 3.14 and Lemma 3.16, there is a ball $\mathcal{B}$ of balanced tax-transfer policies containing 0 such that for each $\mu \in \mathcal{B}$ there is some p with $(p, 1) \in \mathcal{Q}(\omega, \mu)$: that is, $0 \in \operatorname{int} \mathcal{B}$ and $\mathcal{B} \subset \mathfrak{M}_B(\omega)$. Hence, 0 is contained in the relative interior of $\mathfrak{M}_B(\omega)$.

(ii) *Boundedness.* Let ψ and g be the continuous mappings defined in the proof of Proposition 3.16. Let $\bar{\mathcal{P}}$ be the closure of the set of Pareto-optimal allocations $\mathcal{P}$. The mappings g and ψ have continuous extensions from $\mathcal{P}$ to $\bar{\mathcal{P}}$. The image $\psi(\bar{\mathcal{P}})$ is compact and hence bounded. Since we have $\psi(\mathcal{P}) \subset \psi(\bar{\mathcal{P}})$, it follows that $\psi(\mathcal{P})$ is bounded.

(iii) It is well known that the set of Pareto-optimal allocations is arc connected (cf. e.g., Balasko 1979, Appendix 3, pp. 378–379). Hence, $\mathfrak{M}_B(\omega)$ is arc connected as the image by the continuous mapping ψ of the set $\mathcal{P}$. Q.E.D.

3.19. Corollary. Let T be finite. The tax-transfer policy

$$\mu = (\mu_0, \dots, \mu_t, \dots, \mu_T)$$

is bonafide (Definition 2.4) if and only if it is balanced (Definition 3.1). That is, we have $\{\lambda\mu \mid \mu \in \mathfrak{M}_B(\omega) \text{ and } \lambda \in \mathbb{R}_+\} = \{\mu \in \mathbb{R}^{T+1} \mid \sum_{t=0}^{T} \mu_t = 0\}$.

Proof: From Proposition 3.2, we have that if μ is bonafide, then μ is balanced.

From Proposition 3.18, it follows that if μ is balanced so that $\sum_{t=0}^{T} \mu_t = 0$, then for each sufficiently large scalar θ, (μ/θ) belongs to $\mathfrak{M}_B(\omega)$. Hence, from Proposition 2.3, it follows that μ is bonafide (although not necessarily normalized). Q.E.D.

Corollary 3.19 is our central result for the finite-horizon economy. It extends our result for the static economy (see Balasko and Shell 1983, Corollary 3.7) to the perfect-foresight dynamic economy.

That bonafidelity entails balancedness in finite models (including finite overlapping-generations models) is obvious; see Proposition 3.2. The converse is not obvious. Indeed, if our regularity assumptions are relaxed, it is no longer true that balancedness entails bonafidelity. This is illustrated by the next example.

3.20. Example. Let there be two consumers ($t=0,1$), two time periods ($s=1,2$), and only one commodity ($\ell=1$). Assume that consumer 0 has only endowment of and taste for the commodity in period 1 and that consumer 1 has only endowment of and taste for the commodity in period 2. Let $p=(p^1,p^2)\in\mathring{\mathcal{P}}=\{(p^1,p^2)\in\mathbb{R}^2_+\mid p^1+p^2=1\}$, and let $p^m\in\mathbb{R}_+$ be the price of money. Consider the balanced fiscal policy $\mu=(\mu_0,\mu_1)\in\{(\mu_0,\mu_1)\in\mathbb{R}^2\mid\mu_0+\mu_1=0\}$. Equilibrium $(p^1,p^2,p^m)\in\mathring{\mathcal{P}}\times\mathbb{R}_+$ is a solution to the system

$$
\begin{aligned}
(3.21)\quad & \max u_0(x_0^1) \quad \text{subject to } p^1x_0^1\le p^1\omega_0^1+p^m\mu_0,\\
& \max u_1(x_1^2) \quad \text{subject to } p^2x_1^2\le p^2\omega_1^2+p^m\mu_1,\\
& \text{and}\\
& x_0^1\le\omega_0^1 \quad\text{and}\quad x_1^2\le\omega_1^2.
\end{aligned}
$$

The competitive equilibrium allocation is always autarkic, $x_0^1=\omega_0^1$ and $x_1^2=\omega_1^2$, for every $\mu\in\mathbb{R}^2$. Furthermore, p^m must be zero in equilibrium unless $\mu=(0,0)$. Nontrivial tax-transfer policies μ (whether balanced or not) are not bonafide.

In Example 3.20, consumers are not resource related (cf. Arrow and Hahn 1971, Chapter 5, Section 4), but here this is no bar to existence. Competitive equilibrium always exists; indeed, with $p^m=0$, *any* $(p^1,p^2)\in\mathring{\mathcal{P}}$ solves (3.21). Nonetheless, only the trivial fiscal policy is bonafide. The reasoning for this is simple. Nonbalanced policies are not bonafide. Focus then on nontrivial balanced fiscal policies. For one h ($h=0,1$), we have $\mu_h<0$. Consumer h must pay a positive money tax, but he has nothing interesting to exchange with the other consumer, who has received a positive money transfer. Only with $p^m = 0$ can consumer h meet his tax obligation.

Our colleague Dave Cass (in an as-yet unpublished work) has extended the results of this section and of Balasko and Shell (1983). He has weakened our regularity assumptions on individual consumers and added the weaker assumption that consumers are (at least) indirectly resource related. He goes on to establish the equivalence of balanced and bonafide tax-transfer policies in finite economies - whether static or dynamic. (This is to be compared with our Corollary 3.19 in this chapter and Corollary 3.7 in Balasko and Shell 1983.)

In the next proposition, we apply the results of Proposition 3.18 to the set of equilibrium money prices.

3.22. Proposition. Let T be finite and let $\mu = (\mu_0, \dots, \mu_t, \dots, \mu_T)$ be a nontrivial (not necessarily normalized) balanced tax-transfer policy; hence $\sum_{t=0}^{T} \mu_t = 0$. Then the set of equilibrium money prices $\mathcal{P}^m(\omega, \mu)$ (Definition 2.7) is bounded. Furthermore, 0 belongs to $\mathcal{P}^m(\omega, \mu)$ and there is $\bar{p}^m \in \mathbb{R}_{++}$ such that the interval $[0, \bar{p}^m)$ is included in $\mathcal{P}^m(\omega, \mu)$.

Proof: From Proposition 2.8, we know that there is a one-to-one mapping between $\mathfrak{M}_B(\omega) \cap L(\mu)$ and $\mathcal{P}^m(\omega, \mu)$, namely, one in which $p^m \mapsto p^m \mu$. Boundedness of $\mathcal{P}^m(\omega, \mu)$ is a consequence of boundedness of $\mathfrak{M}_B(\omega)$ (Proposition 3.18). Since $0 \in \mathfrak{M}_B(\omega) \cap L(\mu)$, we have $0 \in \mathcal{P}^m(\omega, \mu)$; also see Proposition 2.8.i. Since 0 belongs to the relative interior of $\mathfrak{M}_B(\omega)$ by Proposition 3.18, we have that 0 is not isolated in $\mathcal{P}^m(\omega, \mu)$. Q.E.D.

For a diagrammatic illustration of the relationship between the sets $\mathfrak{M}_B(\omega)$ and $\mathcal{P}^m(\omega, \mu)$, the reader is referred to Balasko and Shell (1983, Figures 1 and 2).

This concludes our analysis of the finite-horizon economy, although we shall employ the results of this section in the following two sections, which are devoted to the infinite-horizon economy.

4 The infinite-horizon model: strongly balanced tax-transfer policies

Let $T = \infty$. The model described in Section 2 then reduces to the model described in Balasko and Shell (1981, especially Sections 2–4). Our goal is to analyze for this infinite-horizon economy the relationship between balanced tax-transfer policies and bonafide tax-transfer policies. For the static economy and the finite-horizon economy, there is no difficulty in defining balancedness of the government's fiscal policy μ. In the infinite-horizon model, the policy $\mu=(\mu_0,\dots,\mu_t,\dots)$ is a sequence in $\mathbb{R}^\infty=\mathfrak{M}$. Not surprisingly, our results are sensitive to the behavior as t becomes large of the sequence $\{\sum_{t=0}^{T}\mu_s\}$.

We begin with our strongest notion of balancedness. We establish that the corresponding tax-transfer policies are always bonafide.

4.1. Definition. The tax-transfer policy $\mu=(\mu_0,\dots,\mu_t,\dots)\in\mathfrak{M}=\mathbb{R}^\infty$ is said to be *strongly balanced* if there is t' $(t'=0,1,\dots)$ with the property $\sum_{s=0}^{t}\mu_s=0$ for $t\ge t'$.

4.2. Remark. If μ is strongly balanced, then obviously $\mu_t=0$ for $t>t'$.

Our first step in the program of establishing that strongly balanced policies are bonafide is to truncate the infinite economy at *period* τ and establish the existence of a normalized equilibrium $(p^m=1)$ for the truncated economy. This is made precise in what follows.

4.3. Definition. The sequence $p=(p^1,\dots,p^\tau,p^{\tau+1},\dots)\in\mathcal{P}\subset(\mathbb{R}^l_{++})^\infty$ is said to be a *normalized τ-equilibrium* if the following τ equations are satisfied:

$$
\begin{aligned}
(4.4)\qquad & f_0^1(p^1,p^1\cdot\omega_0^1+\mu_0)+f_1^1(p^1,p^2,p^1\cdot\omega_1^1+p^2\cdot\omega_1^2+\mu_1)=r^1,\\
& f_1^2(p^1,p^2,p^1\cdot\omega_1^1+p^2\cdot\omega_1^2+\mu_1)+f_2^2(p^2,p^3,p^2\cdot\omega_2^2+p^3\cdot\omega_2^3+\mu_2)=r^2,\\
& \qquad\vdots\qquad\qquad\qquad\qquad\vdots\\
& f_{\tau-1}^\tau(p^{\tau-1},p^\tau,p^{\tau-1}\cdot\omega_{\tau-1}^{\tau-1}+p^\tau\cdot\omega_{\tau-1}^\tau+\mu_{\tau-1})\\
&\qquad +f_\tau^\tau(p^\tau,p^{\tau+1},p^\tau\cdot\omega_\tau^\tau+p^{\tau+1}\cdot\omega_\tau^{\tau+1}+\mu_\tau)=r^\tau,
\end{aligned}
$$

where $f_t^s\in\mathbb{R}^l_{++}$ is the vector of demands of consumer t in period s $(t=0,\dots,\tau;\ s=1,\dots,\tau)$.

4.5. Remark. Definition 4.3 is motivated by Definition 3.1 in Balasko and Shell (1980). Clearly, if $p=(p^1,p^2,\dots,p^t,p^{t+1},\dots)\in\mathcal{P}$ is a normalized τ-equilibrium price sequence, then so is $p'=(p'^1,p'^2,\dots,p'^t,p'^{t+1},\dots)\in\mathcal{P}$ satisfying $p'^t=p^t$ for $t=1,\dots,t+1$. The components $(p^{\tau+2},p^{\tau+3},\dots)$ are indeterminate in a normalized τ-equilibrium.

In what follows, fix t' $(t'=0,1,\dots)$. Consider the strongly balanced tax-transfer policy $\mu^\tau=(\mu_0^\tau,\dots,\mu_t^\tau,\dots)\in\mathbb{R}^\infty$ with the property

(4.6) $\sum_{s=0}^{t} \mu_s^\nu = 0$ for $t \geq t'$.

Then form a sequence $\{\mu^\nu\}$ $(\nu = 1, 2, \ldots)$ of such policies that converges pointwise to the zero sequence as $\nu \to \infty$, that is, $\lim_{\nu \to +\infty} \mu^\nu = 0$.

In the next proposition, we establish that there is a $\bar{\nu}$ with the property that when $\mu = \mu^\nu$ for $\nu \geq \bar{\nu}$, a normalized τ-equilibrium price sequence $p^\nu = ((p^1)^\nu, \ldots, (p^t)^\nu, \ldots) \in \mathcal{P}$ (Definition 4.3) exists for each $\tau \geq t'$.

4.7. Proposition. Fix t' and construct a sequence of (strongly balanced) tax-transfer policies $\{\mu^\nu\}$ that satisfies equation (4.6) for each ν and converges pointwise to zero as $\nu \to +\infty$. There is then a $\bar{\nu}$ with the property that for each $\nu \geq \bar{\nu}$ and each $\tau \geq t'$ there is a normalized τ-equilibrium price sequence $p^\nu \in \mathcal{P}$.

Proof: Append the following equation to the system (4.4):

(4.8) $f_\tau^{\tau+1}(p^\tau, p^{\tau+1}, p^\tau \cdot \omega_\tau^\tau + p^{\tau+1} \cdot \omega_\tau^{\tau+1} + \mu_\tau) = \omega_\tau^{\tau+1}$.

Consider

$$(p^1, \ldots, p^\tau, p^{\tau+1}) \in \{(p^1, \ldots, p^\tau, p^{\tau+1}) \in \mathbb{R}_{++}^{\ell(\tau+1)} \mid p^{1,1} = 1\},$$

which solves the completed system [(4.4) and (4.8)] where $(\mu_0, \ldots, \mu_\tau) = (\mu_0^\nu, \ldots, \mu_\tau^\nu)$ and $\tau \geq t'$ so that $\sum_{t=0}^{t=\tau} \mu_t = 0$.

The completed system [(4.4) and (4.8)] is a $(\tau+1)$-consumer general equilibrium system with a balanced tax-transfer policy. From Lemma 3.16 and Proposition 3.14, we have that there is a $\bar{\nu}$ with the property that for each $\nu \geq \bar{\nu}$ there is some $((p^1)^\nu, \ldots, (p^\tau)^\nu, (p^{\tau+1})^\nu) \in \mathbb{R}_{++}^{\ell(\tau+1)}$ such that $((p^1)^\nu, \ldots, (p^\tau)^\nu, (p^{\tau+1})^\nu, 1) \in \mathcal{Q}(\omega_0, \ldots, \omega_\tau; \mu_0^\nu, \ldots, \mu_\tau^\nu)$. For $\nu \geq \bar{\nu}$, construct the normalized τ-equilibrium price sequence $(p^1, \ldots, p^\tau, p^{\tau+1}, \ldots) \in \mathcal{P}$ by setting $p^t = (p^t)^\nu$ for $t = 1, \ldots, \tau+1$ and arbitrarily choosing $p^t \in \mathbb{R}_{++}^\ell$ for $t > \tau+1$. We have constructed a normalized τ-equilibrium price sequence. Q.E.D.

In what follows, we establish that we can restrict attention to a subset of $\mathcal{P}$ that is compact in the product topology.

4.9. Lemma. Let $\tau \geq t'$ so that $\sum_{t=0}^{\tau} \mu_t^\nu = 0$ for each ν. There are vectors $\bar{\alpha}^t \in \mathbb{R}^\ell$ and $\bar{\beta}^t \in \mathbb{R}^\ell$, $t = 1, \ldots, \tau+1$, and an integer $\bar{\nu}$ with the property that for $\nu \geq \bar{\nu}$ each normalized τ-equilibrium

$$p^\nu = ((p^1)^\nu, (p^2)^\nu, \ldots, (p^\tau)^\nu, (p^{\tau+1})^\nu, \ldots) \in \mathcal{P}$$

satisfies

(4.10) $0<\bar{\alpha}^{t}\leq(p^{t})^{\tau}\leq\bar{\beta}^{t}<+\infty$

for $t=1,\dots,\tau+1$. The bounds are independent of the truncation τ. (The integer t' is fixed.) Furthermore, if $q^{\nu}=(p^{\nu},1)=((p^{1})^{\nu},\dots,(p^{t})^{\nu},\dots;1)\in\mathcal{Q}(\omega_0,\omega_1,\dots;\mu_0^{\nu},\mu_1^{\nu},\dots,\mu_{t'}^{\nu},0,0,\dots)$, then the bounds (4.10) hold for $t=1,2,\dots$.

Proof: For $t=1,\dots,\tau+1$, the existence of the bounds in (4.10) follows from Lemma 3.16 and the proof of Proposition 3.14. Varying $\tau=t',t'+1,\dots$ completes the proof of Lemma 4.9. Q.E.D.

Lemma 4.9 allows us to restrict attention to a compact subset of the set of price sequences.

4.11. Definition. Let $\bar{\mathcal{P}}\subset\mathcal{P}$ be defined by

$$\bar{\mathcal{P}}=\{p=(p^{1},\dots,p^{t},\dots)\in\mathcal{P}\mid 0<\bar{\alpha}^{t}\leq p^{t}\leq\bar{\beta}^{t}<+\infty \text{ for } t=1,2,\dots\},$$

where $\bar{\alpha}^{t}$ and $\bar{\beta}^{t}$ are the bounds defined in Lemma 4.9 [inequalities (4.10)].

Define next $\mathcal{P}(\nu,\tau)$, the set of normalized τ-equilibrium prices sequences in $\bar{\mathcal{P}}$ when the tax transfer policy is μ^{ν}.

4.12. Definition. Fix $\omega\in X$ and let the balanced tax-transfer policy be $\mu^{\nu}=(\mu_0^{\nu},\dots,\mu_t^{\nu},\dots)\in\mathfrak{M}$. Then define $\mathcal{P}(\nu,\tau)$ by

$$\mathcal{P}(\nu,\tau)=\{p\in\bar{\mathcal{P}}\mid p \text{ solves equations (4.4)}\}.$$

Fix t'. By construction, $\{\mathcal{P}(\nu,\tau)\}_{\tau=t'}^{\tau=\infty}$ is a nested, decreasing sequence of nonempty sets. The following two lemmas formalize this idea, which is motivated by Remark 4.2.

4.13. Lemma.

$$\mathcal{P}(\nu,t')\supset\mathcal{P}(\nu,t'+1)\supset\cdots\supset\mathcal{P}(\nu,\tau)\supset\mathcal{P}(\nu,\tau+1)\supset\cdots.$$

Proof: The proof follows immediately from Definitions 4.3 and 4.12. Q.E.D.

4.14. Lemma. Fix t'. There is then a $\bar{\nu}$ such that for $\nu\geq\bar{\nu}$ and $\tau\geq t'$, the set $\mathcal{P}(\nu,\tau)$ is nonempty and compact in the product topology.

Proof: (i) Nonemptiness is an immediate consequence of Proposition 4.7.

(ii) Compactness. We have from Definition 4.11 that $\bar{\mathcal{P}}$ is the infinite product of compact sets. Therefore, by Tychonoff's theorem (see, e.g., Bourbaki 1966, Book I, Section 9.5, Theorem 3, p. 88), $\bar{\mathcal{P}}$ is compact for the product topology. For $\nu \geq \bar{\nu}$ and $\tau \geq t'$, $\mathcal{P}(\nu, \tau)$ is a closed subset of $\bar{\mathcal{P}}$ and hence is compact. Q.E.D.

4.15. Proposition. Let $\mu^\nu \in \mathfrak{M} = \mathbb{R}^\infty$ be a strongly balanced tax-transfer policy. Define $\bar{\nu}$ as in Lemma 4.9. Then for each $\nu \geq \bar{\nu}$, there is a price sequence $p^\nu = ((p^1)^\nu, \dots, (p^t)^\nu, \dots) \in \bar{\mathcal{P}} \subset \mathcal{P}$ such that $(p^\nu, 1) \in \mathcal{Q}(\omega, \mu^\nu)$.

Proof: Let $\mathcal{P}(\nu) \subset (\mathbb{R}^{l}_{++})^\infty$ be defined by $\mathcal{P}(\nu) = \{p \in \bar{\mathcal{P}} \mid (p, 1) \in \mathcal{Q}(\omega, \mu^\nu)\}$. We have $\mathcal{P}(\nu) = \bigcap_{\tau = t'}^{\tau = \infty} \mathcal{P}(\nu, \tau)$. For $\nu \geq \bar{\nu}$, $\mathcal{P}(\nu)$ is equal to the intersection of a nonincreasing sequence of nonempty compact sets (Lemmas 4.13 and 4.14) and is therefore nonempty (see, e.g., Bourbaki 1966, Book I, Section 9.1, pp. 83–84). Q.E.D.

We conclude this section with our central result on strongly balanced tax-transfer policies.

4.16. Proposition. Consider the infinite-horizon ($T = +\infty$) version of the overlapping-generations model presented in Section 2 (see also Balasko and Shell 1981). Each strongly balanced tax-transfer policy is bonafide.

Proof: Fix t' (Definition 4.1). For $\nu \geq \bar{\nu}$ (cf. Lemma 4.9), μ^ν is normalized bonafide (Definition 2.6); that is, $\mu^\nu \in \mathfrak{M}_B(\omega)$ for $\nu \geq \bar{\nu}$, because of Proposition 4.15. But $\{\mu^\nu\}$ is *any* sequence satisfying equation (4.6) and converging pointwise to zero as $\nu \to +\infty$, and hence 0 belongs to the relative interior of $\mathfrak{M}_B(\omega)$ [that is, $0 \in \operatorname{int}(\mathfrak{M}_B(\omega) \cap \{\mu = (\mu_0, \dots, \mu_t, \dots) \mid \sum_{s=0}^{t} \mu_s = 0 \text{ for } t \geq t'\})$]. Thus, by Proposition 2.3, each $\mu = (\mu_0, \dots, \mu_t, \dots)$ satisfying $\sum_{s=0}^{t} \mu_s = 0$ for all $t \geq t'$ is bonafide. Q.E.D.

In the next section, we consider two weaker notions of balancedness. Although these definitions of balancedness seem very natural, neither implies bonafidelity for all economies.

5 The infinite-horizon economy: possible extensions

The central result of Section 4 is that every strongly balanced tax-transfer policy is bonafide. Unless one is willing to severely restrict preferences and endowments, it appears that for the infinite-horizon economy there is no interesting weakening of the definition of balancedness, which necessarily implies bonafidelity. We present here two "conjectured" strengthenings of Proposition 4.16, that is, two weakenings of the hypothesis of Proposition 4.16, and construct counterexamples to each "conjecture."

5.1. Definition. The tax-transfer policy $\mu = (\mu_0, \ldots, \mu_t, \ldots) \in \mathfrak{M} = \mathbb{R}^\infty$ is said to be *asymptotically balanced* if

$$\lim_{t\to\infty} \sum_{s=0}^{t} \mu_s = 0.$$

5.2. "Conjecture." If $\mu \in \mathbb{R}^\infty$ is asymptotically balanced, then μ is bonafide.

Our counterexample to "Conjecture" 5.2 is motivated by the following proposition, which is a simple extension of Proposition 5.8 in Balasko and Shell (1981, p. 129).

5.3. Proposition. Assume that the sequence of aggregate resources $\{r^t\}$ is bounded from above so there is a scalar K $(0 < K < +\infty)$ such that $|r^t| < K$ for $t = 1, 2, \ldots$ ($|\cdot|$ denotes Euclidean norm). Let $\mu = (\mu_0, \ldots, \mu_t, \ldots) \in \mathbb{R}^\infty$ be a bonafide tax-transfer policy admitting a proper competitive equilibrium $q = (p, p^m) \in \mathcal{Q}(\omega, \mu)$, with $p^m \neq 0$. Assume that for some scalar θ $(0 \le \theta < +\infty)$ we have

$$\lim_{t\to\infty} (1+\theta)^t |p^t| = 0.$$

Then μ necessarily satisfies the equation

$$\lim_{t\to\infty} \sum_{s=0}^{t} (1+\theta)^s \mu_s = 0.$$

Proof: If $\mu \in \mathfrak{M}_B(\omega)$ admits the equilibrium $q = (p, 1) \in \mathcal{Q}$, then we have

$$-p^{t+1} \cdot f_{t+1}^{t+1}(p, w_{t+1}) < \sum_{s=0}^{t} \mu_s < p^{t+1} \cdot f_t^{t+1}(p, w_t)$$

for $t = 0, 1, \ldots$ (see Balasko and Shell 1981, Proposition 5.5, p. 126). Hence, we also have

$$\left|\sum_{s=0}^{t-1} \mu_s\right| < \max(|p^t \cdot f_{t-1}^t|, |p^t \cdot f_t^t|) \le |p^t \cdot r^t| \le |p^t||r^t|,$$

and therefore we achieve

$$|p^t| > \frac{1}{|r^t|}\left\|\sum_{s=0}^{t-1}\mu_s\right\| > \frac{1}{K}\left\|\sum_{s=0}^{t-1}\mu_s\right\|,$$

or

$$(1+\theta)^{t-1}K|p^t| > (1+\theta)^{t-1}\left\|\sum_{s=0}^{t-1}\mu_s\right\|.$$

Since by hypothesis the left side of the above inequality approaches zero as t approaches infinity, we have that $(1+\theta)^t\sum_{s=0}^{t}\mu_s$ converges to zero as t approaches infinity. Proposition 2.3 allows this result to be extended to nonnormalized bonafide tax-transfer policies, in which p^m is not equal to unity. Q.E.D.

5.4. Counterexample (to "Conjecture" 5.2). Consider the tax-transfer policy $\mu = (\mu_0, \ldots, \mu_t, \ldots) \in \mathbb{R}^\infty$ given by $\mu_0 = 1/(1-\delta)$ and $\mu_t = -\delta^{t-1}$ for $t = 1, 2, \ldots$. Let δ be a scalar with $0 < \delta < 1$. Then the policy μ is asymptotically balanced, since we have

$$\sum_{t=0}^{\infty}\mu_t = \mu_0 + \sum_{t=1}^{\infty}\mu_t = \frac{1}{1-\delta} - \frac{1}{1-\delta} = 0. \tag{5.5}$$

On the other hand, the public debt is only "retired at infinity" since

$$\sum_{s=0}^{t}\mu_s > 0 \quad \text{for } t = 0, 1, \ldots, \tag{5.6}$$

that is, at each date t ($t = 1, 2, \ldots$), the "supply of outside money" is positive; μ is not strongly balanced.

We proceed to construct an economy for which μ is not a bonafide policy. Take the one-commodity case, $\ell = 1$. Assume stationarity of commodity endowments, that is,

$$\omega_t = (\omega_t^t, \omega_t^{t+1}) = (a, b) \in \mathbb{R}^2_{++}$$

for $t = 1, 2, \ldots,$. We have

$$\begin{aligned} w_t &= p^m\mu_t + p^t\omega_t^t + p^{t+1}\omega_t^{t+1} \\ &= -p^m\delta^{t-1} + p^t a + p^{t+1}b \end{aligned} \tag{5.7}$$

for $t = 1, 2, \ldots$. Define accordingly the nonmonetary equivalent endowment[3] ω_t' for consumer t ($t \geq 1$) by

$$\begin{aligned} \omega_t' &= ((\omega_t^t)', (\omega_t^{t+1})') = (\omega_t^t - (p^m\delta^{t-1}/p^t), \omega_t^{t+1}) \\ &= (a - (p^m\delta^{t-1}/p^t), b). \end{aligned} \tag{5.8}$$

Assume that $q=(p,p^m)=(p^1,\ldots,p^t,\ldots;p^m)\in \mathcal{Q}(\omega,\mu)$ and $p^m\neq 0$. An immediate consequence of equilibrium [cf., e.g., Balasko and Shell 1981, equations (3.4.1), p. 120] is

$$x_t^t = a - \frac{p^m}{p^t}\left[\frac{1}{1-\delta}-\sum_{s=0}^{t-2}\delta^s\right]. \tag{5.9}$$

That is, in equilibrium, consumer t sells period-t commodity to consumer $t-1$ in exchange for all of the money held by consumer $t-1$ (which is equal to $[1/(1-\delta)-\Sigma_{s=0}^{t-2}\delta^s]>0$). We know that

$$\sum_{s=0}^{t}\mu_s=\frac{1}{1-\delta}-\sum_{s=0}^{t-1}\delta^s=\left[\frac{1}{1-\delta}-\sum_{s=0}^{t-2}\delta^s\right]-(\delta)^{t-1}>0,$$

which implies

$$\frac{1}{1-\delta}-\sum_{s=0}^{t-2}\delta^s>\delta^{t-1}. \tag{5.10}$$

Combining equations (5.8)-(5.10) yields

$$\omega_t^t>(\omega_t^t)'>x_t^t, \tag{5.11}$$

which implies that $(\omega_t^t)'>0$, since in equilibrium $x_t^t\in\mathbb{R}_{++}$.

Hence, in equilibrium, (x_t^t,x_t^{t+1}) must belong to $\mathfrak{X}$, a subset of $\mathbb{R}^2_{++}$ defined by

$$\mathfrak{X}=\{(x_t^t,x_t^{t+1})\in\mathbb{R}^2_{++}\mid x_t^t<a \text{ and } x_t^{t+1}>b\}.$$

Without violating our regularity assumptions, we can choose (stationary or nonstationary) utility functions $u_t(\cdot)$ with the property

$$\sup_{(x_t^t,x_t^{t+1})\in\mathfrak{X}}\left(\frac{\partial u_t/\partial x_t^{t+1}}{\partial u_t/\partial x_t^t}\right)<\delta-\epsilon \tag{5.12}$$

for some positive scalar ϵ and for each $t\geq 1$. Thus, in equilibrium, we have

$$0<(p^{t+1}/p^t)<\delta-\epsilon. \tag{5.13}$$

From equations (5.8) and (5.13), we obtain

$$(\omega_t^t)'<a-\frac{p^m(\delta)^{t-1}}{(\delta-\epsilon)^{t-1}}. \tag{5.14}$$

From equation (5.14), we see that for t sufficiently large we have $(\omega_t^t)'<0$, which by equation (5.11) contradicts the equilibrium requirement that $x_t^t\in\mathbb{R}_{++}$.

We have constructed a simple overlapping-generations economy in which the asymptotically balanced (but not strongly balanced) tax-transfer policy μ is not bonafide.

In the following definition, we present our third notion of balancedness for the infinite-horizon economy.

5.15. Definition. The tax-transfer policy $\mu = (\mu_0, \ldots, \mu_t, \ldots) \in \mathbb{R}^\infty$ is said to be *recurrently balanced* if there is a sequence of positive integers $\{t^\nu\}$ approaching $+\infty$ as ν approaches $+\infty$ with the property that for each ν,

$$\sum_{t=0}^{t^\nu} \mu_s = 0.$$

If μ is recurrently balanced (or, alternatively, balanced infinitely often), then for each t^ν (Definition 5.15) there is a corresponding finite-horizon economy that admits a positive equilibrium price of money. This suggests the following "conjecture."

5.16. "Conjecture." If $\mu \in \mathbb{R}^\infty$ is a recurrently balanced tax-transfer policy, then μ is bonafide.

In the following, we sketch a counterexample to "Conjecture" 5.16; we rely on the approach used in Counterexample 5.14.

5.17. Counterexample (to "Conjecture" 5.16). First, we choose $\mu \in \mathbb{R}^\infty$ that is recurrently balanced (Definition 5.15). Let there be a sequence of positive integers $\{t^\nu\}$ such that $t^{\nu+1} > t^\nu$ and

$$\sum_{s=0}^{t^\nu} \mu_s = 0$$

for each t^ν. Let $m^\nu \in \mathbb{R}_+$ be defined by

$$m^\nu = \max_{t^k \in T^\nu} \sum_{s=0}^{t^k} \mu_s, \tag{5.18}$$

where $T^\nu = \{t^\nu, t^\nu + 1, \ldots, t^{\nu+1}\}$. Define t_*^ν as the argmax of the right side of equation (5.18) so that

$$m^\nu = \sum_{s=0}^{t_*^\nu} \mu_s. \tag{5.19}$$

Let δ be a positive fraction. Choose $\mu = (\mu_0, \ldots, \mu_t, \ldots) \in \mathbb{R}^\infty$ to be recurrently balanced (Definition 5.15) with the following additional properties:

(5.20) $\mu_{t_*^\nu+1} = -(\delta)^{t_*^\nu}$

and

(5.21) $m^\nu > \delta^{t_*^\nu}$

for $\nu = 1, 2, \ldots$. By construction, we have $t_*^\nu \in \{t^\nu + 1, \ldots, t^{\nu+1} - 2\}$, that is, $t_*^\nu \neq t^\nu$, $t_*^\nu \neq t^{\nu+1} - 1$, and $t_*^\nu \neq t^{\nu+1}$, for $\nu = 1, 2, \ldots$. Consider next consumer $t_*^\nu + 1$. An immediate consequence of equilibrium in this model is

(5.22) $$(x_{t_*^\nu+1}^{t_*^\nu+1}) = a - \frac{p^m m^\nu}{p^{t_*^\nu+1}}$$

(see Balasko and Shell 1981, equations (3.4.1), p. 120). There are only two consumers alive in period $t_*^\nu + 1$. In equilibrium, consumer $t_*^\nu + 1$ sells period-$(t_*^\nu + 1)$ commodity to consumer t_*^ν in exchange for all his money, m^ν.

As in equation (5.8), we calculate the no-taxation equivalent endowment for consumer $t_*^\nu + 1$,

$$(\omega_{t_*^\nu+1})' = ((\omega_{t_*^\nu+1}^{t_*^\nu+1})', b),$$

where from equation (5.20) we have

(5.23) $$(\omega_{t_*^\nu+1}^{t_*^\nu+1})' = a - \frac{p^m (\delta)^{t_*^\nu}}{p^{t_*^\nu+1}}.$$

Combining equation (5.20)–(5.23) yields

(5.24) $(\omega_{t_*^\nu+1}^{t_*^\nu+1})' > x_{t_*^\nu+1}^{t_*^\nu+1}$.

As before, the utility functions $u_t(\cdot)$ can be chosen so that inequality (5.12) is satisfied on $\mathfrak{X} = \{(x_t^t, x_t^{t+1}) \in \mathbb{R}_{++}^2 \mid x_t^t < a \text{ and } x_t^{t+1} > b\}$ for $t = 1, 2, \ldots$. Thus, for a given $\epsilon > 0$, in equilibrium, commodity prices $\{p^t\}$ must satisfy

(5.25) $0 < (p^{t+1}/p^t) < \delta - \epsilon$

for $t = 1, 2, \ldots$. But $p^1 = 1$, and thus for ν sufficiently large, $(\omega_{t_*^\nu+1}^{t_*^\nu+1})'$ is negative by equations (5.23) and (5.25). Therefore, using equation (5.24), we have $(x_{t_*^\nu+1}^{t_*^\nu+1})$ is also negative for ν sufficiently large, which is a contradiction to the equilibrium conditions (Definition 2.1).

We have constructed a simple overlapping-generations economy in which the recurrently balanced (but not strongly balanced) tax-transfer policy μ is not bonafide.

5.26. Remark. Counterexample 5.17 shows that not all recurrently balanced tax-transfer policies are bonafide. Close reading of the argument in the counterexample [see especially equations (5.20) and (5.21)] shows that if we restrict attention to recurrently balanced policies $\mu = (\mu_0, \ldots, \mu_1, \ldots) \in \mathbb{R}^\infty$ with the additional property $m^\nu \to 0$ as $\nu \to +\infty$, we can still construct economies for which μ is not bonafide. Policies with the property that the limit superior of the outside money supply is zero are not necessarily bonafide even if we restrict attention to recurrently balanced tax-transfer policies.

Finally, we remind the reader of the well-known fact that there are economies for which nonbalanced tax-transfer policies are bonafide.

5.27. Definition. A tax-transfer policy $\mu = (\mu_0, \ldots, \mu_t, \ldots) \in \mathfrak{M} = \mathbb{R}^\infty$ is said to be *passive* if $\mu_t = 0$ for $t \geq 1$.

5.28. Observation. Consider the passive tax-transfer policy

$$\mu = (1, 0, \ldots, 0, \ldots) \in \mathbb{R}^\infty.$$

The policy μ is obviously not balanced in any relevant sense, since $\sum_{s=0}^{t} \mu_s = 1$ for $t = 0, 1, \ldots$. Nonetheless, there exist (stationary) economies for which μ is bonafide.

Proof: See, for example, Samuelson (1958, especially pp. 477–482) and Gale (1973, Figure 1, p. 20). Q.E.D.

6 Concluding remarks

Perfectly foreseen, nondistorting government fiscal policy matters in the overlapping-generations economy. Taxes and transfers are the means of effecting interpersonal and intergenerational redistribution. The only neutrality result relevant to fiscal policy in the overlapping-generations model is the independence of the timing of taxes and tranfers for a *given* individual consumer (see our note 1).

This chapter can be taken in part as a response to the current (neo-Ricardian) fixation with repaying the public debt. Debt retirement is central to fiscal policy in finite perfect-foresight models; see, for example, Balasko and Shell (1983) and Section 3 of this chapter. (There have also been claims that the centrality of debt retirement extends to the popular macroeconomic model based on a finite number of infinitely lived families.[4]) For the infinite overlapping-generations model, Samuelson (1958) has taught us that there are environments in which fiscal policy can be effective without debt retirement; see Observation 5.28. The Samuelson example answers only one side of the debt retirement issue. In Sections 4 and 5, we investigate the other side. We establish that if the debt is retired in finite time, then fiscal policy can be effective (for every environment); see Proposition 4.16. On the other hand, we also show that if debt retirement in the infinite-horizon economy does not proceed at a sufficiently rapid rate for the given environment, then fiscal policy is necessarily ineffective; see Proposition 5.3, Counterexamples 5.4 and 5.17, and Remark 5.26.

NOTES

1 It is implicitly assumed that each consumer faces perfect borrowing and lending markets during his lifetime. An immediate consequence of overall equilibrium is that the *present* price of money, $p^{m,t}$, is independent of date t, i.e., $p^{m,t} = p^m \in \mathbb{R}_+$ for $t = 1, 2, \ldots, T+1$. It also follows from equilibrium considerations that consumer t's opportunities depend only on his lifetime transfer μ_t (and are independent of the time profile of his transfers). These "arbitrage" arguments are carefully spelled out in Balasko and Shell (1981, Sections 2–3). One interpretation of our model is that the markets for inside money are perfect and that the price of inside money is positive even when the price of outside money is zero. At any rate, by assumption no Clower (or Clower-like) constraints are binding in this chapter.

2 A representation of this important isomorphism is given in Balasko and Shell (1981, Fig. 7.1, Erratum), in which the emphasis is on the infinite horizon. See also Balasko and Shell (1983, Figs. 2 and 3), in which the (finite) static economy is analyzed.

3 Compare the definition of nonmonetary equivalent endowment, ω_t', [equation (5.8)] with that of adjusted endowment, $\tilde{\omega}_t$, in the associated no-taxation economy [equations (3.3)]. Notice that $\tilde{\omega}_t \in \mathbb{R}^{t(T+1)}$ is constructed from ω_t by adjusting commodity 1 *in period* 1, whereas $\omega_t' \in \mathbb{R}^2$ is constructed from ω_t by adjusting the single commodity *in period t*.

4 We have seen some of these claims, but we have seen no proof of them. It is our belief that the model with infinitely lived consumers is in fact more complicated than most users of the model imagine.

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