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MARKET UNCERTAINTY: SUNSPOT EQUILIBRIA IN IMPERFECTLY COMPETITIVE ECONOMIES*

by

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1. Introduction and Summary.

Businessmen are responsible for making economic decisions in the face of uncertainty. Some of this uncertainty is the result of uncertainty about economic fundamentals -- tastes, endowments, and production possibilities. The weather, for example, affects the economy through its influence on crop yields and through its influence on the final demand for umbrellas and swimsuits. This is the type of uncertainty, uncertainty transmitted to the economy through uncertainty about the fundamental parameters of the economy, which is modelled by conventional general-equilibrium theory.

There is another, significant source of uncertainty which businessmen actually face. This is market uncertainty, uncertainty generated within the economy itself, uncertainty about the economic outcomes (such as prices) given the fundamental parameters. Businessmen wonder whether the economy will be healthy or unhealthy, whether consumers' confidence will be high or low, whether credit will be loose or tight, whether the dollar will be strong or weak, whether potential rivals will hold back or enter, whether prices for their products will be high or low, whether prices for their factors will be low or high, and so forth. It seems to us that most businessmen quite properly worry more about the uncertainty of the outcome of the market process than they do about the uncertainty of the fundamental parameters.1/

The formal modelling of market uncertainty has until recently lagged far behind the modelling of uncertainty which is transmitted to the economy through uncertainty about its fundamental parameters. The recent work on so-called "Sunspot Equilibrium" introduced by Cass and Shell (reported in Shell [17] and Cass and Shell [6]) is meant to advance our understanding of market uncertainty. We now know that, in rational-expectations models, the competitive-equilibrium allocation of resources can be random, even if the economic fundamentals are immune from random disturbances.

The present paper represents an extension of the analysis of market uncertainty to economies in which competition is imperfect. The scope of the effects of market uncertainty is richer and of greater policy interest in the noncompetitive environment. In this environment, economic actors are uncertain as to whether markets will be thick (with many trades taking place) or thin (with few trades taking place), or even whether these markets will be open or closed. In attempting an assessment of market thickness, an economic actor must judge the confidence of other economic actors. A wide range of such beliefs is rational (i.e., self-justifying). If, for example, demand in a particular market (or overall) is weak, then in response supply is weak, which in turn justifies (establishes the rationality of) the weak demand.

We build on the familiar price-formation model of Shapley and Shubik (cf. Shubik [18], Shapley and Shubik [16], and Shapley [15]). We use this particular type of model as a stage for our analysis because by now it is very well-known and substantially elaborated 2/. Our basic analysis could have been based on any one of several alternative general-equilibrium formulations of imperfect competition, e.g., the Cournot-Nash version used by Heller in [9].

The (certainty) Market Game is introduced in Section 2. All money is inside money; there is no government debt. Players do not face liquidity or credit constraints. The autarkic allocation is always an equilibrium outcome. There is a wide range of equilibrium outcomes in which each of the commodity markets is open.

In Section 3, the Market Game is modified to allow extrinsic uncertainty to play a role in the allocation of resources. We refer to the extrinsic random variable as "sunspot activity", since it has no effect on economic fundamentals, including tastes and endowments. The spot commodity markets are the same as in Section 2, but there are also securities for transferring income across states of nature. We refer to the modified game as the Securities Game. Our securities markets are analogous to those put forward by Arrow [1]. The Securities Game can also be thought of as a noncompetitive variant of the sunspot model of Cass and Shell [6], although here no exogenous restrictions are placed on market participation.

The equilibria of the Market Game reappear as Nonsunspot Equilibria to the Securities Game. We show that, in addition, there are always Sunspot Equilibria to the Securities Game. For some of the Sunspot Equilibria, the securities market is closed; for some, the securities market is open but inactive (no net trades); and for others, the securities market is active. Some Sunspot Equilibrium allocations to the Securities Game are Correlated Equilibrium allocations to the Market Game (in the sense of Aumann [2,3]); others are not.

In Section 4, we analyze the large economy, which is the limit of the replicated Securities Game. In the limit economy, the market power of individual consumers vanishes; the economy is competitive but with the possibility that some markets are closed. There are many rational-expectations equilibria in the limit economy, some of which are Sunspot Competitive Equilibria; the rest are Nonsunspot Competitive Equilibria.

2. Pure certainty: The Market Game.

There are $\ell+1$ goods: ℓ commodities (or consumption goods), indexed by $i = 1,\ldots,\ell$ and $j = 1,\ldots,\ell$, and money. There are neither taxes nor transfers, so all money is "inside money", representing the private debt of the consumers. There are n consumers (or traders), indexed by $h = 1,\ldots,n$ and $k = 1,\ldots,n$. Consumer h is endowed with a positive amount of commodity i, ω_h^i, for $i = 1,\ldots,\ell$. If we denote by ω_h the endowment vector $(\omega_h^1,\ldots,\omega_h^i,\ldots,\omega_h^\ell)$, then we have $\omega_h \in \mathbb{R}_{++}^\ell$ for $h = 1,\ldots,n$.

There are ℓ trading posts. For each commodity, there is a single trading post on which the commodity is exchanged for money. Consumer h supplies a nonnegative quantity of commodity i, q_h^i, at trading post i. He also supplies a nonnegative quantity of money, b_h^i, at trading post i. We say that q_h^i is his <u>offer</u> (of commodity i) and that b_h^i is his (money) <u>bid</u> (for commodity i). Let $b_h = (b_h^1,\ldots,b_h^i,\ldots,b_h^\ell)$ and $q_h = (q_h^1,\ldots,q_h^i,\ldots,q_h^\ell)$ denote (respectively) his <u>bids</u> and his <u>offers</u>. Offers must be made in terms of the physical commodities. Hence, offers cannot exceed endowments, i.e., we have $q_h^i \leq \omega_h^i$ for $i = 1,\ldots,\ell$. The <u>strategy set</u> S_h of consumer h is then given by

$$S_h = \{(b_h,q_h) \in \mathbb{R}_+^{2\ell} \mid q_h \leq \omega_h\} .$$

The trading process is simple. The total amount of commodity i which is offered, $\sum_{k=1}^{k=n} q_k^i$, is allocated to consumers in proportion to their shares of the bids for commodity i. Consumer h's share of the bids at post i is $(b_h^i / \sum_{k=1}^{k=n} b_k^i)$. Thus, the gross receipts of commodity i for consumer h are

$$\frac{b_h^i}{\sum_{k=1}^{k=n} b_k^i} \sum_{k=1}^{k=n} q_k^i$$

for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. If all bids at post i are zero, the ratio $(b_h^i / \sum_{k=1}^{k=n} b_k^i)$ is equal to 0/0 and would appear to be indeterminate. We assume, however, that if there are no bids on post i all offers on this post are "lost", i.e., no commodity is delivered. Thus, we take the fraction $(b_h^i / \sum_{k=1}^{k=n} b_k^i)$ to be zero if there are no positive bids at post i.

At trading post i, the money from bids, $\sum_{k=1}^{k=n} b_k^i$, is allocated to consumers in proportion to their offers of commodity i. Consumer h's share of the offers at post i is $(q_h^i / \sum_{k=1}^{k=n} q_k^i)$. Thus the gross money receipts on post i for consumer h are

$$\frac{q_h^i}{\sum_{k=1}^{k=n} q_k^i} \sum_{k=1}^{k=n} b_k^i$$

for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. If all offers of commodity i are zero, the ratio $(q_h^i / \sum_{k=1}^{k=n} q_k^i)$ is equal to $0/0$. We assume that if there are no offers on post i, all money bids on the post are "lost". Thus, we take the fraction $(q_h^i / \sum_{k=1}^{k=n} q_k^i)$ to be zero if there are no positive offers on post i.

Consumers do not face liquidity constraints, i.e., constraints which restrict their debt issuance on any given market or proper subset of markets. Each consumer does face a single overall budget constraint, which he must meet or be punished. He is required to finance his bids (for commodities) by his offers (of commodities). The <u>budget constraint</u> for consumer h is

$$\text{(2.1)} \qquad \sum_{j=1}^{j=\ell} \{(q_h^j / \sum_{k=1}^{k=n} q_k^j) \sum_{k=1}^{k=n} b_k^j\} \geq \sum_{j=1}^{j=\ell} b_h^j ,$$

for $h = 1,\ldots,n$. The right-hand side of Inequality (2.1) is the sum of the dollars delivered by h in the form of bids to the trading posts. The left-hand side is the sum of the dollars delivered to h from the trading posts in payment for his commodity offers. The consumer is punished if he issues more money debt than he collects.[3/]

Let x_h^i denote the consumption of commodity i by consumer h, and let $x_h = (x_h^1, \ldots, x_h^i, \ldots, x_h^\ell)$ be his consumption vector. Assume that consumer k chooses the strategy $(b_k, q_k) \in \mathbb{R}_+^{2\ell}$ for $k = 1, \ldots, n$; then the consumption of consumer h is given by

$$(2.2) \quad \begin{cases} x_h^i = \omega_h^i - q_h^i + \dfrac{b_h^i}{\sum_{k=1}^{k=n} b_k^i} \sum_{k=1}^{k=n} q_k^i & \text{if (2.1) is satisfied, and} \\ x_h^i = 0 & \text{if (2.1) is not satisfied} \end{cases}$$

for $i = 1, \ldots, \ell$ and $h = 1, \ldots, n$. Failure to meet budget constraint (2.1) leads to confiscation of all of the consumer's goods.[4/]

The consumption set of consumer h is the nonnegative orthant $\{x_h \mid x_h \in \mathbb{R}_+^\ell\}$. His utility function, u_h, is strictly increasing, smooth, and strictly concave on the strictly positive orthant $\mathbb{R}_{++}^\ell$. Also, the closure in $\mathbb{R}^\ell$ of each indifference surface from $\mathbb{R}_{++}^\ell$ is contained in $\mathbb{R}_{++}^\ell$. (This last assumption allows us to avoid some pesky boundary solutions.) The boundary of the consumption set, $(\mathbb{R}_+^\ell \setminus \mathbb{R}_{++}^\ell)$, is also the indifference surface of least utility, so that (i) if we have $x_h \in (\mathbb{R}_+^\ell \setminus \mathbb{R}_{++}^\ell)$ and $y_h \in (\mathbb{R}_+^\ell \setminus \mathbb{R}_{++}^\ell)$, then we also have $u_h(x_h) = u_h(y_h) = u_h(0)$, and (ii) if we have $x_h \in (\mathbb{R}_+^\ell \setminus \mathbb{R}_{++}^\ell)$ and $y_h \in \mathbb{R}_{++}^\ell$, then we also have $u_h(y_h) > u_h(x_h) = u_h(0)$.

We have specified the <u>strategy sets</u> S_h, the <u>outcomes</u> x_h (through Equation (2.2)), and the <u>payoffs</u> $u_h(x_h)$ for the Market Game Γ. We adopt the standard concept of <u>Nash Equilibrium</u> (NE).

Let $\sigma_h = (b_h, q_h)$ be a strategy in S_h. Define the set S by $S = S_1 \times \ldots \times S_h \times \ldots \times S_n \subset (\mathbb{R}_+^{2\ell})^n$. Consider the strategies $\sigma =$

$(\sigma_1,\ldots,\sigma_h,\ldots,\sigma_n) = ((b_1,q_1),\ldots,(b_h,q_h),\ldots,(b_n,q_n)) \in S$, $(\sigma|\sigma_h') =$ $(\sigma_1,\ldots,\sigma_{h-1},\sigma_h',\sigma_{h+1},\ldots,\sigma_n) = ((b_1,q_1),\ldots,(b_{h-1},q_{h-1}),(b_h',q_h'),$ $(b_{h+1},q_{h+1}),\ldots,(b_n,q_n)) \in S$, and $\sigma_{-h} = ((b_1,q_1),\ldots,(b_{h-1},q_{h-1}),$ $(b_{h+1},q_{h+1}),\ldots,(b_n,q_n)) \in S_1 \times\ldots\times S_{h-1} \times S_{h+1} \times \ldots \times S_n \subset (\mathbb{R}_+^{2\ell})^{n-1}$. From Equation (2.2), we see that x_h^i is a function of the b's and q's, so that the outcome can be written as a function of the strategies σ, namely $x_h(\sigma)$.

2.3 Definition. A Nash Equilibrium strategy to the Market Game Γ is a $\sigma \in S$ with the property

$$u_h(x_h(\sigma)) = \max_{\sigma_h' \in S_h} \{u_h(x_h(\sigma|\sigma_h'))\}$$

for $h = 1,\ldots,n$. The corresponding NE allocation is $x(\sigma) =$ $(x_1(\sigma),\ldots,x_h(\sigma),\ldots,x_n(\sigma)) \in \mathbb{R}_+^{\ell n}$.

We next establish that consumer h's optimal response σ_h to the (equilibrium or disequilibrium) strategies of others is "individually rational".

2.4. Lemma. Let σ_h be consumer h's best response to the strategies σ_{-h} in the market game Γ, i.e.,

$$u_h(x_h(\sigma)) = \max_{\sigma_h' \in S_h} \{u_h(x_h(\sigma|\sigma_h')\}.$$

Then, we have

$$u_h(x_h(\sigma)) \geq u_h(\omega_h) .$$

Proof: If consumer h selects the trivial strategy $\sigma_h' = (b_h', q_h') = (0,0)$, then we have $x_h(\sigma|\sigma_h') = \omega_h$. Hence, if σ_h is the best response to σ_{-h}, it follows that $u_h(x_h(\sigma)) \geq u_h(x_h(\sigma|\sigma')) = u_h(\omega_h)$. $\square$

This shows that in equilibrium no consumer is punished, since we have $u_h(x_h) \geq u_h(\omega_h) > u_h(0)$, thus justifying the logic behind Equation (2.2).

We next show that there is always a very simple NE for the market game Γ. Later, we show that there are also other NE for Γ.

2.5. Lemma. Let $\sigma = (\sigma_1, \ldots, \sigma_h, \ldots, \sigma_n)$ be the trivial vector defined by $\sigma_h = (b_h, q_h) = (0,0)$ for $h = 1, \ldots, n$. Then σ is a NE strategy for Γ. The associated NE allocation $x(\sigma) = (x_1(\sigma), \ldots, x_h(\sigma), \ldots, x_n(\sigma))$ is defined by $x_h(\sigma) = \omega_h$ for $h = 1, \ldots, n$.

Proof: Obvious. $\square$

It is well known that autarkic NE can be trivially exhibited in this type of market game. Autarkic NE will play an important role in our analysis of Sunspot Nash Equilibria (cf. Section 3 of this paper). They also represent an extreme example of a general phenomenon, which is readily explained by this model. Nash market games do not exhibit the same degree of coordination of plans which is present in Walrasian models. In Nash models, supplies can be limited by an insufficiency of aggregate demands. Here supplies are zero because demands are zero. The zero supplies in turn justify the zero demands. The circle is closed. We shall elaborate on this important aspect of the model later.

In the autarkic NE, there are no bids and no offers. All markets are closed. We shall study existence of NE in which markets are open. We need definitions of an open (and a closed) market.

2.6. Definition. Let $\sigma = ((b_1,q_1),\ldots,(b_h,q_h),\ldots,(b_n,q_n))$ be a NE vector of strategies in the Market Game Γ. We say that market j is closed (resp. open) if $\sum_{k=1}^{k=n} b_k^j = 0$ (resp. $\sum_{k=1}^{k=n} b_k^j > 0$).

2.7. Lemma. Let $\sigma = ((b_1,q_1),\ldots,(b_h,q_h),\ldots,(b_n,q_n))$ be a NE vector of strategies in the Market Game Γ. Market j is closed (resp. open) if and only if $\sum_{k=1}^{k=n} q_k^j = 0$ (resp. $\sum_{k=1}^{k=n} q_k^j > 0$).

Proof: If $\sum_{k=1}^{k=n} b_k^j = 0$, then it follows from the definition of NE and the monotonicity of utility functions that $q_h^j = 0$ for $h = 1,\ldots,n$. Furthermore, if $\sum_{k=1}^{k=n} q_k^j = 0$, then it follows that $b_h^j = 0$ for $h = 1,\ldots,n$. $\square$

We see immediately how NE are affected by beliefs about markets. If people believe that market j is closed, it will be closed. It is a rational self-fulfilling belief.

Next we justify a normalization for the vector of bids, $b \in \mathbb{R}_+^{\ell n}$.

2.8. Lemma. If it is not the case that all markets are closed, we can without loss in generality, restrict the vector of bids b to be in the unit simplex, i.e., we have $\sum_j \sum_k b_k^j = 1$ or, simply, $b \in \Delta^{\ell n}$.

Proof: From Definition (2.6) and Lemma (2.7) and the fact that some market is open, we have $\sum_j \sum_k b_k^j > 0$. From Equation (2.2), we have that x_h^i is homogeneous of degree zero in b. Hence, without loss of generality, we can impose the restriction $b \in \Delta^{\ell n}$. $\square$

Since $\sum_k q_k^i$ is measured in units of commodity i it serves as a good measure of "market thickness". When $\sum_k q_k^i$ is zero, market i is closed. When $\sum_k q_k^i$ is small relative to $\sum_k \omega_k^i$, we are tempted to say that market i is thin. For larger values of $\sum_k q_k^i$, on the order of $\sum_k \omega_k^i$, we are tempted to say that market i is thick.

What beliefs about market "thickness" are self-justifying? We have already seen that if consumers believe a market to be closed, their beliefs are justified. In order to further pursue the analysis of market thickness, we next consider Offer-Constrained Market Games and Bid-Constrained Market Games. We begin with definitions of the strategy sets for these games.

2.9. Definition. (i) Fix $q_h = \bar{q}_h \in \mathbb{R}_{++}^{\ell}$, where we have $0 < \bar{q}_h^i \leq \omega_h^i$ for $i = 1,\ldots,\ell$. Let $S_h(\bar{q}_h) = \{(b_h,q_h) \mid b_h \in \mathbb{R}_+^{\ell} \text{ and } q_h = \bar{q}_h\}$ be the <u>offer-constrained strategy set</u> for consumer h. Let $S(\bar{q}) = S_1(\bar{q}_1) \times\ldots\times S_h(\bar{q}_h) \times\ldots\times S_n(\bar{q}_n)$, where $\bar{q} = (\bar{q}_1,\ldots,\bar{q}_h,\ldots,\bar{q}_n) \in \mathbb{R}_{++}^{n\ell}$. (ii) Fix $b_h = \bar{b}_h \in \mathbb{R}_{++}^{\ell}$. Let $S_h(\bar{b}_h) = \{(b_h,q_h) \mid 0 \leq q_h^i \leq \omega_h^i \text{ for } i = 1,\ldots,\ell \text{ and } b_h = \bar{b}_h\}$ be the <u>bid-constrained strategy set</u> for consumer h. Let $S(\bar{b}) = S_1(\bar{b}_1) \times\ldots\times S_h(\bar{b}_h) \times\ldots\times S_n(\bar{b}_n)$, where $\bar{b} = (\bar{b}_1,\ldots,\bar{b}_h,\ldots,\bar{b}_n) \in \mathbb{R}_{++}^{n\ell}$.

We next define the offer-constrained and bid-constrained games, and the corresponding NE strategies and allocations.

2.10. Definition. (i) The <u>Offer-Constrained Market Game</u> $\Gamma(\bar{q})$ is the same as the Market Game Γ except that the strategy set S is replaced by the offer-constrained strategy set $S(\bar{q})$ (Definition (2.9)). (ii) The Bid-Constrained Market Game $\Gamma(\bar{b})$ is the same as the Market Game Γ except that the strategy set S is replaced by the bid-constrained strategy set $S(\bar{b})$ (Definition (2.9)).

A NE strategy for the Offer-Constrained Game $\Gamma(\bar{q})$ is a $\sigma \in S(\bar{q})$ with the property

$$u_h(x_h(\sigma)) = \max_{\sigma'_h \in S_h(\bar{q}_h)} \{u_h(x_h(\sigma|\sigma'_h))\}$$

for $h = 1,\ldots,n$, where $x_h(\sigma)$ and $x_h(\sigma|\sigma'_h)$ are defined by Equation (2.2). The vector $x(\sigma) = (x_1(\sigma),\ldots,x_h(\sigma),\ldots,x_n(\sigma)) \in \mathbb{R}^{\ell n}_{++}$ is the corresponding NE allocation.

A NE strategy for the Bid-Constrained Game $\Gamma(\bar{b})$ is a $\sigma \in S(\bar{b})$ with the property

$$u_h(x_h(\sigma)) = \max_{\sigma'_h \in S_h(\bar{b}_h)} \{u_h(x_h(\sigma|\sigma'_h)\}$$

for $h = 1,\ldots,n$, where $x_h(\sigma)$ and $x_h(\sigma|\sigma')$ are defined by Equation (2.2). The vector $x(\sigma) = (x_1(\sigma),\ldots,x_h(\sigma),\ldots,x_n(\sigma)) \in \mathbb{R}^{\ell n}_{++}$ is the corresponding NE allocation.

How are the NE of Γ, the NE of $\Gamma(\bar{q})$, and the NE of $\Gamma(\bar{b})$ related? An answer to this question is provided in the following proposition.

2.11. Proposition. (i) Let $\sigma = \{(b_h,q_h)\}_{h=1}^{h=n}$ be a NE strategy for the Offer-Constrained Market Game $\Gamma(\bar{q})$. If bids are strictly positive, i.e., if we have $b_h \in \mathbb{R}^{\ell}_{++}$ for $h = 1,\ldots,n$, then σ is also a NE strategy for the

Market Game Γ. (ii) Let $\sigma = \{(b_h, q_h)\}_{h=1}^{h=n}$ be a NE strategy for the Bid-Constrained Market Game $\Gamma(\bar{b})$. If offers are strictly positive, i.e., we have $q_h \in \mathbb{R}^{\ell}_{++}$ for $h = 1,\ldots,n$, then σ is also a NE strategy for the Market Game Γ.

<u>**Proof**</u>: Form the Lagrangian expression

$$\Lambda_h(b_h, q_h, \theta_h, \eta_h, \lambda_h) = u_h(\ldots, \omega_h^i - q_h^i + (b_h^i \textstyle\sum_k q_k^i)/\sum_k b_k^i, \ldots) +$$

$$(2.12) \qquad \sum_{j=1}^{j=\ell} \theta_h^j b_h^j + \sum_{j=1}^{j=\ell} \eta_h^j q_h^j +$$

$$\lambda_h \sum_{j=1}^{j=\ell} \{[(q_h^j \textstyle\sum_k b_k^j)/\sum_k q_k^j] - b_h^j\},$$

where θ_h^j, η_h^j, and λ_h are nonnegative (scalar) Kuhn-Tucker-Lagrange multipliers.[5/]

(i) Let $\hat{\sigma} = \{(\hat{b}_h, \hat{q}_h)\}_{h=1}^{h=n}$ be a NE strategy for $\Gamma(\bar{q})$ and assume that $\hat{b}_h \in \mathbb{R}^{\ell}_{++}$. It follows from $\bar{q} \in \mathbb{R}^{\ell n}_{++}$ that Budget Constraint (2.1) is binding for consumer h. Fix $q_h = \bar{q}_h$ in Λ_h and differentiate the Lagrangian expression (2.12) with respect to b_h^i, which yields the first-order condition

$$(2.13) \qquad \frac{\partial u_h}{\partial x_h^i} = \frac{\lambda_h (\sum_{k=1}^{k=n} b_k^i)^2 (\sum_{k \neq h} \bar{q}_k^i)}{(\sum_{k=1}^{k=n} \bar{q}_k^i)^2 (\sum_{k \neq h} b_k^i)}$$

because at the optimum θ_h^i must equal zero because $\hat{b}_h^i > 0$. Since this is a problem in concave programming the set of first-order conditions defined by (2.13) (for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$) characterize the optimal-response bids $\hat{b}$ given the offers $\bar{q}$.

We next show that the offers $\bar{q}$ are optimal responses to the bids $\hat{b}$. Fixing $b_h = \hat{b}_h$ in Λ_h in Expression (2.12) and then differentiating with respect to q_h^i yields the first-order condition

$$(2.14) \qquad \left[\frac{\partial u_h}{\partial x_h^i} \frac{-\sum_{k\neq h}\hat{b}_k^i}{\sum_k \hat{b}_k^i}\right] + \eta_h^i + \frac{\lambda_h(\sum_k \hat{b}_k^i)(\sum_{k\neq h} q_k^i)}{(\sum_k q_k^i)^2} = 0 .$$

Since $\bar{q}_h^i > 0$, we have $\eta_h^i = 0$, so Equation (2.14) reduces to

$$(2.15) \qquad \frac{\partial u_h}{\partial x_h^i} = \frac{\lambda_h(\sum_k \hat{b}_k^i)^2 \, (\sum_{k\neq h} q_h^i)}{(\sum_k q_k^i)^2 \, (\sum_{k\neq h} \hat{b}_k^i)} .$$

Compare Equations (2.13) and (2.15). The bids $\hat{b} \in \mathbb{R}_{++}^{\ell n}$ are NE responses to the offers $\bar{q}$ if and only if the first-order constraint is satisfied. But if $\{(\hat{b}_h, \bar{q}_h)\}$ solves the first-order equations defined in (2.13), then $\{(\hat{b}_h, \bar{q}_h)\}$ must also solve the first-order equations defined in (2.15) for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. We have shown that $\{(\hat{b}_h, \hat{q}_h)\}_{h=1}^{h=n}$ is a NE for Γ.

(ii) In like fashion, we can show that if σ is a NE to $\Gamma(\bar{b})$ with $q_h \in \mathbb{R}_{++}^{\ell}$ for $h = 1,\ldots,n$, then σ is also a NE to Γ. $\square$

Obviously not all beliefs about market thickness are self-justifying. If a relatively small offer, $\bar{q}_h^i$, is imposed on consumer h, he might have desired to sell more of commodity i given the bids and offers of others, i.e., the constraint $q_h^i \leqq \bar{q}_h^i$ would be binding. The constraint $b_h^i \geq 0$ could then also become binding: he might want to bid a negative amount to make up for his meager offer, $\bar{q}_h^i$. On the other hand, (as the next proposition establishes), if all offers are sufficiently large, then they are self-justifying. Closed markets are self-justifying, thick markets are self-justifying, but some thin markets are not self-justifying.

2.16. Proposition. Let $\sigma = \{(b_h, q_h)\}_{h=1}^{h=n}$ be a NE to the Offer-Constrained Market Game $\Gamma(\bar{q})$. There is a positive scalar ζ such that, if

$$q_h^i = \bar{q}_h^i > \omega_h^i - \zeta$$

for $h = 1,\ldots,n$ and $i = 1,\ldots,\ell$, then σ is also a NE to the Market Game Γ.

Proof: Consider consumer h. Offers are fixed at $\bar{q} = (\bar{q}_1,\ldots,\bar{q}_h,\ldots,\bar{q}_n) \in \mathbb{R}_{++}^{\ell n}$. He takes the bids of others $b_{-h} = (b_1,\ldots,b_{h-1},b_{h+1},\ldots,b_n) \in \mathbb{R}_+^{\ell(n-1)}$ as fixed. He can guarantee his endowment by the bidding strategy given by

$$b_h^i = \frac{\bar{q}_h^i}{\sum_{k\neq h} \bar{q}_k^i} \sum_{k\neq h} b_k^i , \tag{2.17}$$

for $i = 1,\ldots,\ell$; cf. Equation (2.2). Hence, we have

$$u_h(x_h(\sigma)) \geq u_h(\omega_h) .$$

Thus, we have that if $x = (x_1,\ldots,x_h,\ldots,x_n)$ is a NE allocation, then we know that $x_h \in K_h \subset \mathbb{R}_{++}^{\ell}$, where

$$(2.18) \qquad K_h = \{ x_h \in \mathbb{R}^{\ell}_{++} \mid u_h(x_h) \geq u_h(\omega_h) \text{ and } x_h \leq \sum_{k=1}^{k=n} \omega_k \} .$$

The set K_h is convex, compact, and by our assumption on the closure of indifference surfaces, bounded away from the axes. Hence there is a positive scalar ζ^i_h with the property that for each $x_h = (x^1_h,\ldots,x^i_h,\ldots,x^{\ell}_h) \in K_h$, we have $x^i_h > \zeta^i_h$ for $i = 1,\ldots,\ell$. Therefore, for each NE allocation $x = (x_1,\ldots,x_h,\ldots,x_n)$ there is a positive scalar ζ such that $x^i_h > \zeta$ for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$.

Assume that $\bar{q}^i_h > \omega^i_h - \zeta$, where ζ is the positive scalar just constructed. Consumer h must make a positive bid on market i, $b^i_h > 0$, or else the consumption vector x_h would not belong to the set K_h. Hence, if $\sigma = \{(b_h,q_h)^{h=n}_{h=1}$ is a NE for $\Gamma(\bar{q})$ all bids must be positive, i.e., $b_h \in \mathbb{R}^{\ell}_{++}$ for $h = 1,\ldots,n$. Therefore, by Proposition (2.11), σ is also a NE for the (unconstrained) Market Game Γ. $\square$

Next, we define an interior NE strategy vector and then study its welfare and existence properties.

2.19. Definition. The strategy $\sigma = \{(b_h,q_h)\}^{h=n}_{h=1} \in S$ is said to be an interior Nash Equilibrium to the market game Γ if σ is a NE for Γ (Definition (2.3)) in which each of the ℓ markets is open (Definition (2.6)). The corresponding allocation $x(\sigma) = (x_1(\sigma),\ldots,x_h(\sigma),\ldots,x_n(\sigma)) \in \mathbb{R}^{\ell n}_{++}$ is called an interior NE allocation of Γ.

2.20. Proposition. An interior NE allocation of Γ is autarkic (i.e., $x_h = \omega_h$, for $h = 1,\ldots,n$) if and only if the endowment vector $\omega = (\omega_1,\ldots,\omega_h,\ldots,\omega_n)$ is Pareto optimal. Furthermore, if ω is not Pareto optimal, an interior NE allocation of Γ, $x = (x_1,\ldots,x_h,\ldots,x_n)$, must satisfy $u_h(x_h) \geq u_h(\omega_h)$, with strict inequality for at least one h, $h = 1,\ldots,n$.

Proof: Let $x = (x_1,\ldots,x_h,\ldots,x_n)$ be an interior NE allocation which is autarkic, i.e., $x = \omega = (\omega_1,\ldots,\omega_h,\ldots,\omega_n)$. The interior first-order condition,

$$\text{(2.21)} \qquad \frac{\partial u_h(x_h)/\partial x_h^i}{\partial u_h(x_h)/\partial x_h^j} = \frac{\sum\limits_{k\neq h} q_k^i}{\sum\limits_{k\neq h} b_k^i}\left[\frac{\sum\limits_{k=1}^{k=n} b_k^i}{\sum\limits_{k=1}^{k=n} q_k^i}\right]^2 \frac{\sum\limits_{k\neq h} b_k^j}{\sum\limits_{k\neq h} q_k^j}\left[\frac{\sum\limits_{k=1}^{k=n} q_k^j}{\sum\limits_{k=1}^{k=n} b_k^j}\right]^2$$

for $i,j = 1,\ldots,\ell$, (along with equality in Budget-Constraint (2.1)) is necessary and sufficient for consumer h's utility to be optimized given the strategies of the other consumers. Since $x_h = \omega_h$, we must have

$$\frac{b_h^i}{q_h^i} = \frac{\sum\limits_{k=1}^{k=n} b_k^i}{\sum\limits_{k=1}^{k=n} q_k^i}$$

for $i = 1,\ldots,\ell$, because of Equation (2.2). Hence, First-Order Condition (2.21) yields

$$\text{(2.22)} \qquad \frac{\partial u_h(\omega_h)/\partial x_h^i}{\partial u_h(\omega_h)/\partial x_h^j} = \frac{\sum\limits_{k=1}^{k=n} b_k^i}{\sum\limits_{k=1}^{k=n} q_k^i}\left[\frac{\sum\limits_{k=1}^{k=n} q_k^i}{\sum\limits_{k=1}^{k=n} b_k^j}\right],$$

for $i,j = 1,\ldots,\ell$. Since the right-hand side of Equation (2.22) is independent of h, all consumers have the same marginal rates of substitution, establishing that the allocation is Pareto optimal.

Consumer h, given the strategies of others, can always guarantee his endowment by setting $q_h^i = 0$ and $b_h^i = 0$ for $i = 1,\ldots,\ell$. Thus, if $x = (x_1,\ldots,x_h,\ldots,x_n)$ is a NE allocation, we have $u_h(x_h) \geq u_h(\omega_h)$ for $h = 1,\ldots,n$.

It follows that, if ω is Pareto optimal, we have $u_h(x_h) = u_h(\omega_h)$ for $h = 1,\ldots,n$. Since u_h is strictly increasing and strictly concave on $\mathbb{R}_{++}^{\ell}$, if ω is Pareto optimal then the NE allocation must be autarkic, i.e., we must have $x = \omega$.

Assume next that the endowment vector, $\omega = (\omega_1,\ldots,\omega_h,\ldots,\omega_n)$, is not Pareto optimal. Let $x = (x_1,\ldots,x_h,\ldots,x_n)$ be an interior NE allocation for Γ. We have already established that (i) $x_h \neq \omega_h$ for at least one h, and (ii) $u_h(x_h) \geq u_h(\omega_h)$ for all h. Since u_h is strictly increasing and strictly quasi-concave, we know that the strict inequality $u_h(x_h) > u_h(\omega_h)$ holds for at least one h. $\square$

2.23. Proposition. There is an interior NE strategy $\sigma = \{(b_h,q_h)\}_{h=1}^{h=n} \in S$ for the Market Game Γ.

Proof: We adopt the following plan:

First, we specify that offers are equal to endowments, i.e., we have $q_h^i = \omega_h^i \in \mathbb{R}_{++}$ for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. Hence, we will have constructed from the original Market Game Γ the Offer-Constrained Market Game $\Gamma(\bar{q})$, where $\bar{q} = \omega \in \mathbb{R}_{++}^{\ell n}$. We know that if $\{(b_h,\omega_h)\}_{h=1}^{h=n}$ is a NE strategy for $\Gamma(\bar{q})$, where $\bar{q} = \omega$, then $\{(b_h,\omega_h)\}_{h=1}^{h=n}$ must be an interior NE strategy for $\Gamma(\bar{q})$, and by Proposition (2.16), $\{(b_h,\omega_h)\}_{h=1}^{h=n}$ is an interior NE strategy for the (unconstrained) Market Game Γ.

Second, we seek a vector of bids $b = (b_1,\ldots,b_h,\ldots,b_n)$ in the unit simplex $\Delta^{\ell n}$ with the property that $\{(b_h,\omega_h)\}_{h=1}^{h=n}$ is a NE for $\Gamma(\bar{q})$, where $\bar{q} = \omega$. One looks for a continuous mapping of the simplex $\Delta^{\ell n}$ into itself, a fixed point of which is an (interior) NE of the offer-constrained game.

Let $y_h' = ((y_h^1)',\ldots,(y_h^i)',\ldots,(y_h^\ell)') \in \mathbb{R}^\ell_{++}$ be a variable representing the bids of consumer h. Choose $b \in \Delta^{\ell n}$ and construct $(b|y_h')$, where $(b|y_h')$ is b but with b_h replaced by y_h'. The endowment vector is fixed and the vector of offers is fixed, so from Equation (2.2), consumer h's utility varies only with changes in bids. Therefore, we can express utility as a function of the bids, $u_h(b)$. Let $y_h = (y_h^1,\ldots,y_h^i,\ldots,y_h^\ell) \in \mathbb{R}^\ell_+$ be the value of y_h' which maximizes $u_h(b|y_h')$ subject to $(y_h^i)' \geq 0$ for $i = 1,\ldots,\ell$.

One is then <u>tempted</u> to map $b \mapsto b'$ as follows

$$(b_h^i)' = \frac{y_h^i}{\sum_{j=1}^{\ell}\sum_{k=1}^{n} y_k^j} ,$$

for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. The approach does not work, because the mapping described above is not well-defined. Let $b_k^i = 0$ for some i and each $k \neq h$, so that $\sum_{k\neq h} b_k^i = 0$. If consumer h bids $(y_h^i)' > 0$, he would expect to receive all of commodity i, $\sum_{k=1}^{k=n} \omega_k^i$. For each positive $(y_h^i)'$ there is a smaller bid which is preferred by consumer h, but $y_h^i = 0$ is clearly not an optimal bid for consumer h. In order to avoid this difficulty, we replace $b_k^i \in \mathbb{R}_+$ with $\beta_k^i \in \mathbb{R}_{++}$, where

$$\beta_k^i = \max(b_k^i, \varepsilon) \tag{2.24}$$

and ε is a positive scalar. Let $\beta_h = (\beta_h^1, \ldots, \beta_h^i, \ldots, \beta_h^\ell) \in \mathbb{R}_{++}^\ell$ for $h = 1, \ldots, n$ and $\beta = (\beta_1, \ldots, \beta_h, \ldots, \beta_n) \in \mathbb{R}_{++}^{\ell n}$. Also let $(\beta|z_h')$ be the vector β but with β_h replaced by $z_h' = ((z_h^1)', \ldots, (z_h^i)', \ldots, (z_h^\ell)') \in \mathbb{R}_+^\ell$ and let $\beta_{-h} = (\beta_1, \ldots, \beta_{h-1}, \beta_{h+1}, \ldots, \beta_n) \in \mathbb{R}_{++}^{\ell(n-1)}$. Let $z_h = (z_h^1, \ldots, z_h^i, \ldots, z_h^\ell)$ be the value of z_h' which maximizes $u_h(\beta|z_h')$. Then define $(b_h^i)'$ by

$$(b_h^i)' = \frac{z_h^i}{\sum_{j=1}^{\ell} \sum_{k=1}^{n} z_k^j}, \tag{2.25}$$

for $i = 1, \ldots, \ell$ and $h = 1, \ldots, n$. Equation (2.25) defines a mapping from $\Delta^{\ell n} \to \Delta^{\ell n}$, which takes the point $b = (b_1^1, \ldots, b_h^i, \ldots, b_n^\ell)$ to the point $b' = ((b_1^1)', \ldots, (b_h^i)', \ldots, (b_n^\ell)')$. We proceed to establish that for ε sufficiently small there is a (strictly positive) vector $b^* = ((b_1^1)^*, \ldots, (b_h^i)^*, \ldots, (b_n^\ell)^*) \in \Delta^{\ell n}$ which is a fixed point to the mapping $b \mapsto b'$ defined by Equation (2.25). The purpose of the following is to put bounds on the denominator of the right-hand side of Equation (2.25), which will be used in establishing the continuity of this mapping for ε sufficiently small.

<u>Claim 1</u>: Consider bids $b \in \Delta^{\ell n}$ and choose ε to satisfy

$$0 < \varepsilon < (1/\ell n). \tag{2.26}$$

Then there are positive scalars $\underline{\theta}$ and $\overline{\theta}$, <u>independent of b and ε</u>, with the property

$$\underline{\theta} \leq \sum_{j=1}^{j=\ell} \sum_{k=1}^{k=n} z_k^j \leq \bar{\theta} \,, \tag{2.27}$$

where the z_k^j's are the maximizing responses to the "adjusted bids" of others, β_{-k}.

__Proof of Claim 1:__ Consider $\beta_h = (\beta_h^1,\ldots,\beta_h^i,\ldots,\beta_h^\ell)$, consumer h's optimizing response, in the game $\Gamma(\bar{q})$ with $\bar{q} = \omega$, to the adjusted bids β_{-h}. From Inequality (2.1), β_h must satisfy

$$\sum_{j=1}^{\ell} z_h^j \leq A\Big(\sum_{k\neq h} \sum_{j=1}^{\ell} \beta_k^j + \sum_{j=1}^{\ell} z_h^j\Big) \,, \tag{2.28}$$

where

$$1 > A = \max_{\substack{i = 1,\ldots,\ell \\ h' = 1,\ldots,n}} \left(\frac{\omega_{h'}^i}{\sum_{k=1}^{n} \omega_k^i} \right) > 0 \,.$$

We have $\beta_k^j \leq b_k^j + \varepsilon$ and $\sum_{k=1}^{n} \sum_{j=1}^{\ell} b_k^j = 1$; hence we have

$$1 + \ell n\varepsilon \geq \sum_{k=1}^{n} \sum_{j=1}^{\ell} \beta_k^j > \sum_{k\neq h} \sum_{j=1}^{\ell} \beta_k^j \,. \tag{2.29}$$

Combining Inequalities (2.26), (2.28), and (2.29), yields

$$\sum_{j=1}^{j=\ell} z_h^j < 2A/(1-A) \,.$$

Since we have $0 < A < 1$, we have established the existence of the upper bound $\bar{\theta}$.

Because $b \in \Delta^{\ell n}$, there must be some consumer h and some commodity i such that

$$\sum_{k \neq h} \beta_k^i \geq \sum_{k \neq h} b_k^i \geq (1/\ell n) \ .$$

Assume that $z_h = (z_h^1, \ldots, z_h^i, \ldots, z_h^\ell)$ is h's optimizing response to the "adjusted bids" β_{-h}. The resulting consumption by consumer h would be given by

$$x_h^i = \left[\frac{z_h^i}{z_h^i + \sum_{k \neq h} \beta_k^i}\right] \sum_{k=1}^{n} \omega_k^i \ ,$$

for $i = 1, \ldots, \ell$, from Equation (2.2) where we set $q_k^i = \omega_k^i$. Therefore, we have

$$x_h \leq \sum_{k=1}^{n} \omega_k \in \mathbb{R}_{++}^{\ell} \ ,$$

even if the β_{-h} are disequilibrium bids. Furthermore, consumer h can always "defend his endowment" ω_h by the strategy described in Equation (2.17) in the proof of Proposition (2.16), and hence we have

$$u_h(x_h) \geq u_h(\omega_h) \ ,$$

even if the β_{-h} are disequilibrium bids. Thus, consumption x_h must lie in the compact, convex set K_h defined in Equation (2.18). K_h is also -- by the assumption on the closure of indifference surfaces -- bounded away from the axes. Thus, there is a positive scalar ζ such that $x_h^i > \zeta$ for

$h = 1,\ldots,n$ and $i = 1,\ldots,\ell$. Hence, we have

$$\frac{z_h^i \sum_{k=1}^{n} \omega_k^i}{z_h^i + \sum_{k \neq h} \beta_k^i} \geq \zeta$$

and thus we have

$$z_h^i \geq \frac{(\zeta/\ell n)}{\sum_{k=1}^{n} \omega_k^i} .$$

This establishes the existence of the lower bound $\underline{\theta}$. The proof of Claim 1 is complete.

Claim 2: Choose ε to satisfy Inequality (2.26). Then the mapping $b \mapsto b'$ defined by Equations (2.24)-(2.25) has a fixed point, $b^* \in \Delta^{\ell n}$.

Proof of Claim 2: "Adjusted bids" β are obviously single-valued and continuous in bids b for every ε. Since u_h is strictly quasi-concave, z_h is single-valued and continuous in β (and therefore in b). If we have $0 < \varepsilon < (1/\ell n)$, then by Claim 1 the mapping $b \mapsto b'$ is single valued and continuous. The mapping takes $\Delta^{\ell n}$ into itself. An application of Brouwer's fixed-point theorem completes the proof.

Claim 3: Let z_h be consumer h's optimal response to the bids b_{-h} (or to "adjusted" bids β_{-h}) and let x_h be the corresponding consumption plan of consumer h. The marginal rates of commodity substitution satisfy

$$\alpha \leq \frac{\partial u_h(x_h)/\partial x_h^i}{\partial u_h(x_h)/\partial x_h^j} \leq \gamma \, , \tag{2.30}$$

where α and γ are positive scalars, for $i,j = 1,\ldots,\ell$ and $h = 1,\ldots,n$.

Proof of Claim 3: The result follows from the fact that we have $x_h \in K_h$, a compact set, for $h = 1,\ldots,n$, and that the marginal rates of substitution are continuous functions of x_h.

2.31 Definition. Let M be given by

$$M = \frac{(1+\bar{\theta})^2 M_1}{\alpha[\min(\underline{\theta},1)]}, \tag{2.32}$$

where

$$M_1 = \max_{h,i,j} \left[\frac{\sum_{k\neq h} \omega_k^i}{\left(\sum_{k=1}^{n} \omega_k^i\right)^2} \; \frac{\left(\sum_{k=1}^{n} \omega_k^j\right)^2}{\sum_{k\neq h} \omega_k^j} \right], \tag{2.33}$$

α is defined in Inequality (2.30), and $\underline{\theta}$ and $\bar{\theta}$ are defined in Inequality (2.27).

In the next claim, we show there is a lower bound to the sum of the fixed-point adjusted bids on any market. Intuitively, we exclude the case where the price on some market is arbitrarily small.

Claim 4: Let ε satisfy Inequality (2.26). Let $b^* = ((b_1^1)^*,\ldots,(b_h^i)^*,\ldots,(b_n^\ell)^*)$ be the fixed-point vector of bids, the existence of which was established in Claim 2 above, and let $\beta^* = ((\beta_1^1)^*,\ldots,(\beta_h^i)^*,\ldots,(\beta_n^\ell)^*)$ be the corresponding vector of adjusted bids defined by $(\beta_h^i)^* = \max((b_h^i)^*,\varepsilon)$.

Then we have

$$\sum_{k=1}^{n} (\beta_k^i)^* \geq 1/\ell nM \quad , \tag{2.34}$$

for $i = 1,\ldots,\ell$, where M is given in Definition (2.31).

Proof of Claim: Assume the contrary. In particular, assume that on some market (call it market i) we have

$$\sum_{k=1}^{n} (\beta_k^i)^* < 1/\ell nM \quad . \tag{2.35}$$

There must be some consumer (call him consumer h) whose fixed-point bid, $(b_h^i)^*$, is less than or equal to the sum of the augmented bids of the others on market i. Since in response to β_{-h}^*, consumer h will choose a consumption basket in K_h (Definition (2.18)), and since the first-order utility-maximizing conditions are necessary and sufficient, we have

$$\frac{\left[\dfrac{z_h^i + \sum_{k\neq h} (\beta_k^i)^*}{\sum_{k=1}^{n} \omega_k^i}\right]^2 \left(\dfrac{\sum_{k\neq h} \omega_k^i}{\sum_{k\neq h} (\beta_k^i)^*}\right)}{\left[\dfrac{z_h^j + \sum_{k\neq h} (\beta_k^j)^*}{\sum_{k=1}^{n} \omega_k^j}\right]^2 \left(\dfrac{\sum_{k\neq h} \omega_k^j}{\sum_{k\neq h} (\beta_k^j)^*}\right)} \geq \alpha \tag{2.36}$$

for $i,j = 1,\ldots,\ell$, because the left-hand side of Inequality (2.36) is the marginal rate of substitution

$$\frac{\partial u_h(x_h)/\partial x_h^i}{\partial u_h(x_h)/\partial x_h^j} \quad ;$$

cf. Claim 3 above. Combining Inequality (2.36) with Equation (2.33) yields

$$\text{(2.37)} \qquad \frac{M_1 \sum_{k\neq h} (\beta_k^j)^*}{\sum_{k\neq h} (\beta_k^i)^*} \left[\frac{z_h^i + \sum_{k\neq h} (\beta_k^i)^*}{z_h^j + \sum_{k\neq h} (\beta_k^j)^*} \right] \geq \alpha \quad .$$

From Equation (2.25), the fact that b^* is a fixed point, and Inequality (2.27), we have

$$z_h^i \leq \bar{\theta}(b_h^i)^* \; .$$

But market i was chosen so that

$$\sum_{k\neq h} (\beta_k^i)^* < (1/\ell nM),$$

by Inequality (2.35). Hence, we have

$$\text{(2.38)} \qquad z_h^i + \sum_{k\neq h} (\beta_k^i)^* < \bar{\theta}(b_h^i)^* + (1/\ell nM) < (\frac{1}{\ell nM})(\bar{\theta}+1) \quad .$$

By similar reasoning, we get

$$\text{(2.39)} \qquad z_h^j + \sum_{k\neq h} (\beta_k^j)^* > \underline{\theta}(b_h^j)^* + \sum_{k\neq h} (\beta_k^j)^* > \frac{1}{\ell n} [\min(\underline{\theta},1)] \quad .$$

Substituting from Inequalities (2.38) and (2.39) into Inequality (2.37) yields

$$(2.40)\qquad \frac{M_1(\bar\theta + 1)}{M[\min(\underline{\theta},1)]}\left(\frac{\bar\theta(b_h^i)^*}{\sum\limits_{k\neq h}(\beta_k^i)^*} + 1\right) > \alpha \ .$$

Consumer h was chosen because his bid is less than or equal to $\sum\limits_{k\neq h}(\beta_k^i)^*$, so using Inequality (2.40), we have

$$(2.41)\qquad \frac{(1+\bar\theta)^2 M_1}{\alpha[\min(\underline{\theta},1)]} > M \ ,$$

which is a contradiction to Equation (2.32), in which M is defined. This contradiction completes the proof of Claim 4.

<u>**Claim 5**</u>: Choose ε to satisfy Inequality (2.26) and

$$(2.42)\qquad \varepsilon < \zeta/(\ell n M\bar\theta \max_j (\sum_{k=1}^{n} \omega_k^j)) \ ,$$

where ζ is the lower bound on x_h^i $(i = 1,\ldots,\ell;\ h = 1,\ldots,n)$ which follows from $x_h \in K_h$ $(h = 1,\ldots,n)$ (cf. Equation (2.18)). Let $b^* = ((b_1^1)^*,\ldots,(b_h^i)^*,\ldots,(b_n^\ell)^*)$ be the fixed-point vector of bids, the existence of which was established in Claim 2 above. Then we have

$$(2.43)\qquad (b_h^i)^* > \varepsilon$$

for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$.

<u>**Proof of Claim 5**</u>: Assume the contrary. Then there is some consumer (call him consumer h) and some market (call it market i) with the property

$$(2.44)\qquad (b_h^i)^* \leq \varepsilon,$$

so that we have

(2.45)
$$(\beta_h^i)^* = \varepsilon.$$

Given the adjusted bids of the others, β_{-h}, h's responses z_h must be consistent with $x_h \in K_h$ (cf. Equation (2.18)), so we must have

(2.46)
$$\zeta < \frac{z_h^i \max_j \left[\sum_{k=1}^{n} \omega_k^j\right]}{\sum_{k=1}^{n} (\beta_k^j)^*} .$$

Using Inequalities (2.27; Claim 1) and (2.34; Claim 4) and Stipulation (2.45) in Inequality (2.46) yields

(2.47)
$$\zeta < \frac{\varepsilon\bar{\theta} \max_j \sum_{k=1}^{n} \omega_k^j}{1/\ell nM} .$$

There is no positive ε which satisfies both Inequality (2.42) and Inequality (2.47). This is a contradiction to Assumption (2.44). The proof of Claim 5 is complete.

Claim 6: Choose ε to satisfy Inequalities (2.26) and (2.42). Let $b^* = ((b_1^1)^*,\ldots,(b_h^i)^*,\ldots,(b_n^\ell)^*)$ be the fixed point, the existence of which is established in Claim 2. Then $\sigma = \{(b_h,q_h)\}_{h=1}^{h=n} = \{(b_h^*,\omega_h)\}_{h=1}^{h=n}$ is a NE strategy for the Offer-Constrained Game $\Gamma(\bar{q})$ where $\bar{q} = \omega$.

Proof of Claim 6: From Claim 2, Claim 5, and Equations (2.24)-(2.25), we have

$$(2.48) \qquad (b_h^i)^* = \frac{z_h^i(b^*)}{\sum_{k=1}^{n} \sum_{j=1}^{\ell} z_k^j(b^*)} ,$$

where, for example, $z_h^i(b^*)$ is consumer h's response to $(b_{-h})^* = (\beta_{-h})^*$. We shall show that the denominator in Equation (2.48) is equal to unity. This will be accomplished in two steps.

<u>Step 1</u>: Assume that

$$\sum_{j=1}^{\ell} \sum_{k=1}^{n} z_k^j(b^*) > 1.$$

Then we have $(b_h^i)^* < z_h^i(b^*)$ for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. Given $(b_{-h})^*$, consumer h desires to increase each of his bids. Hence, at bids b^*, there is slack in budget Constraint (2.1); namely, we have

$$\sum_{j=1}^{\ell} (b_h^j)^* < \sum_{j=1}^{\ell} \left[\frac{\omega_h^j}{\sum_{k=1}^{n} \omega_k^j} \sum_{k=1}^{n} (b_k^j)^* \right] .$$

Summing the above over consumers yields

$$(2.49) \qquad \sum_{j=1}^{\ell} \sum_{k=1}^{n} (b_k^j)^* < \sum_{j=1}^{\ell} \sum_{k=1}^{n} \left[\frac{\omega_k^j}{\sum_{k'=1}^{n} \omega_{k'}^j} \sum_{k'=1}^{n} (b_{k'}^j)^* \right] .$$

The right-hand side of Equation (2.49) can be simplified to become $\sum_j \sum_k (b_k^j)^*$, which yields a contradiction. Hence, we have $\sum_j \sum_k z_k^j(b^*) \leq 1$.

Step 2: Assume that

$$\sum_{j=1}^{\ell} \sum_{k=1}^{n} z_k^j(b^*) < 1.$$

Then we would have $(b_h^i)^* > z_h^i(b^*)$ for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. The only way this could occur is if the bids b^* were not consistent with Budget Constraint (2.1), i.e.,

$$\sum_{j=1}^{\ell} (b_h^j)^* > \sum_{j=1}^{\ell} \left[\frac{\omega_k^j}{\sum_{k=1}^{n} \omega_k^j} \sum_{k=1}^{n} b_k^j \right] .$$

Summing the above over consumers and simplifying as before yields

$$\sum_j \sum_k (b_k^j)^* > \sum_j \sum_k (b_k^j)^* , \tag{2.50}$$

which is a contradiction. It follows that

$$\sum_{j=1}^{\ell} \sum_{k=1}^{n} z_k^j(b^*) = 1$$

and hence

$$(b_h^i)^* = z_h^i(b^*),$$

for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. We have established that b^* are NE bids for offers $\bar{q} = \omega$ in the game $\Gamma(\bar{q})$. The proof of Claim 6 is complete.

Proof of Proposition (2.23): We know from Claim 2 that there is a fixed point b^* to the mapping defined by Equations (2.24)-(2.25) if ε is chosen to satisfy Inequality (2.26). If ε also satisfies Inequality (2.42), we have from Claims 5 and 6 that $z_h^i(b^*) = (b_h^i)^* = (\beta_h^i)^*$ for $i = 1,\ldots,\ell$ and $h = 1,\ldots,n$. Hence, the strategy $\sigma = \{(b_h,q_h)\}_{h=1}^{h=n} = \{(b_h^*,\omega_h)\}_{h=1}^{h=n}$ is a NE to the Offer-Constrained Market Game $\Gamma(\bar{q})$ where $\bar{q} = \omega$. From Proposition (2.16), we also have that σ is a NE strategy for the (unconstrained) Market Game Γ. This completes the proof of Proposition (2.23).[6/] $\square$

3. Extrinsic uncertainty: The Securities Game.

In this section, we introduce purely extrinsic uncertainty. The state of nature can be thought of as given by the level of "sunspot activity". By definition, the fundamentals of the economy -- here, tastes and endowments -- are unaffected by sunspot activity. Spot market trading is the same as described in Section 2, but there are now securities markets in which consumers are able to hedge against the potential economic effects of sunspots.

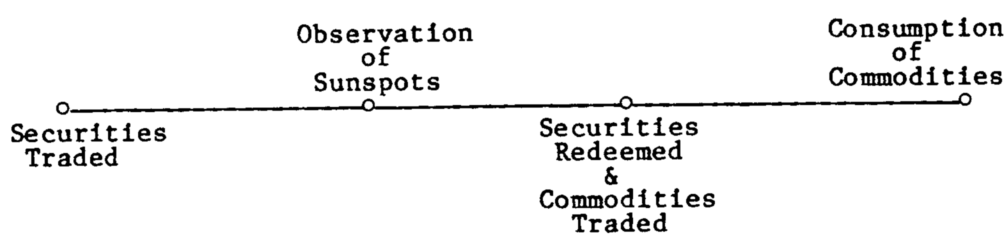

Figure 1

Figure 1 is our time line. Each of the n consumers is alive and active during the entire period. There are r states of nature indexed by s. Let $x_h(s) = (x_h^1(s),\ldots,x_h^i(s),\ldots,x_h^\ell(s)) \in \mathbb{R}_{++}^\ell$ be consumer h's consumption basket if state s occurs $(s = 1,\dots,r$ and $h = 1,\dots,n)$ and define $\tilde{x}_h = (x_h(1),\dots,x_h(s),\dots,x_h(r)) \in \mathbb{R}^{\ell r}_{++}$ and $\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_h,\dots,\tilde{x}_n) \in \mathbb{R}^{\ell rn}_{++}$. Consumer h has the strictly concave von Neumann-Morgenstern utility function v_h defined by

$$v_h(\tilde{x}_h) = \sum_{s=1}^{r} \pi(s)u_h(x_h(s)) ,$$

where u_h is the utility function described in Section 2, $\pi(s)$ is the (objective) probability of the occurrence of state s, $0 < \pi(s) < 1$, $\sum_{s=1}^{s=r} \pi(s) = 1$. Let $\omega_h(s) = (\omega_h^1(s),\dots,\omega_h^i(s),\dots,\omega_h^\ell(s)) \in \mathbb{R}^\ell_{++}$ be consumer h's endowments in state s, and define $\tilde{\omega}_h = (\omega_h(1),\dots,\omega_h(s),\dots,\omega_h(r)) \in \mathbb{R}^{\ell r}_{++}$ and $\tilde{\omega} = (\tilde{\omega}_1,\dots,\tilde{\omega}_h,\dots,\tilde{\omega}_n) \in \mathbb{R}^{\ell rn}_{++}$. Since uncertainty is purely extrinsic, we have

$$\omega_h(s) = \omega_h$$

for $s = 1,\dots,r$ and $h = 1,\dots,n$, where ω_h is the certainty endowment vector introduced in Section 2.

The new feature is the securities market, composed of r trading posts, one for each state of nature. Bids are denominated in "general monetary units", but offers are made in state-specific units of account. After the state of nature s is observed, consumers trade on the spot market, composed as in Section 2 of ℓ posts, one for each commodity. Let $b_h^i(s)$ and $q_h^i(s)$ be, respectively, the bid and the offer of consumer h on spot market trading post i given that state of nature s has occurred. Let $b_h^m(s)$ and $q_h^m(s)$ be, respectively, the bid and the offer of consumer h on security market s. Define $\tilde{b}_h$ and $\tilde{q}_h$ by $\tilde{b}_h = (b_h^1(1),\dots,b_h^i(1),\dots,b_h^\ell(1),\dots,b_h^1(s),\dots,b_h^i(s),\dots,b_h^\ell(s),$ $\dots,b_h^1(r),\dots,b_h^i(r),\dots,b_h^\ell(r);\ b_h^m(1),\dots,b_h^m(s),\dots,b_h^m(r)) \in \mathbb{R}^{r(\ell+1)}_{+}$ and

$\tilde{q}_h = (q_h^1(1),\ldots,q_h^i(1),\ldots,q_h^\ell(1),\ldots,q_h^1(s),\ldots,q_h^i(s),\ldots q_h^\ell(s),\ldots,q_h^1(r),$ $\ldots,q_h^i(r),\ldots,q_h^\ell(r);\ q_h^m(1),\ldots,q_h^m(s),\ldots,q_h^m(r)) \in \mathbb{R}_+^{r(\ell+1)}$. Also define the strategy $\tilde{\sigma} = (\tilde{\sigma}_1,\ldots,\tilde{\sigma}_h,\ldots,\tilde{\sigma}_n)$ by $\tilde{\sigma}_h = (\tilde{b}_h,\tilde{q}_h)$. Then the strategy set $\tilde{S}_h$ for consumer h in the securities game $\tilde{\Gamma}$ is given by

$$(3.1)\quad \tilde{S}_h = \{\ \tilde{\sigma}_h \in \mathbb{R}_+^{2r(\ell+1)} \mid q_h^i(s) \leq \omega_h^i \text{ for } i = 1,\ldots,\ell \text{ and } s = 1,\ldots,r\ \}.$$

Define $\tilde{\sigma}$ and $\tilde{S}$ by $\tilde{\sigma} = (\tilde{\sigma}_1,\ldots,\tilde{\sigma}_h,\ldots,\tilde{\sigma}_n) \in \tilde{S}_1 \times\ldots\times \tilde{S}_h \times\ldots\times \tilde{S}_n = \tilde{S}$. Also define $\tilde{\sigma}_{-h}$ by $\tilde{\sigma}_{-h} = (\tilde{\sigma}_1,\ldots,\tilde{\sigma}_{h-1},\tilde{\sigma}_{h+1},\ldots,\tilde{\sigma}_n) \in \tilde{S}_1 \times\ldots\times \tilde{S}_{h-1} \times \tilde{S}_{h+1} \times\ldots\times \tilde{S}_n$ and $(\tilde{\sigma}|\tilde{\sigma}_h')$ by $(\tilde{\sigma}|\tilde{\sigma}_h') = (\tilde{\sigma}_1,\ldots,\tilde{\sigma}_{h-1},\tilde{\sigma}_h',\tilde{\sigma}_{h+1},\ldots,\tilde{\sigma}_n) \in \tilde{S}$.

There are two markets. The securities market, which meets before the state of nature, s, is observed and the spot commodities market, which meets after the state of nature s is observed. Consumer h must satisfy two constraints, one for each market; if either one or both are not satisfied, consumer h is punished. The securities-market constraint is:

$$(3.2.i)\qquad \sum_{s=1}^{s=r} b_h^m(s) \leq \sum_{s=1}^{s=r}\left[q_h^m(s)\, \frac{\sum_{k=1}^{k=n} b_k^m(s)}{\sum_{k=1}^{k=n} q_k^m(s)} \right],$$

i.e., the sum of the securities-market bids in "general dollars" (the left-hand side of Inequality (3.2.i)) must be no greater than the sum of the revenue in "general dollars" from the sales of securities[7/] (the right-hand side of Inequality (3.2.i)). Purchases of securities are financed by the sales of securities. A single unit of security s pays one unit of account in state s and zero otherwise. Security s can be thought of as state-s money, or state-s dollars, dollars accepted in state s and only in state s.

If consumer h had neither bought nor sold securities, he would have faced the commodity-market budget constraint

$$\sum_{j=1}^{j=\ell} b_h^j(s) \leq \sum_{j=1}^{\ell} \left[q_h^j(s) \frac{\sum_{k=1}^{k=n} b_k^j(s)}{\sum_{k=1}^{k=n} q_k^j(s)} \right]$$

for $s = 1,\ldots,r$. Without any securities income or obligations, he must finance his bids for commodities in state-s dollars (the left-hand side of the above inequality) from his receipts in state-s dollars based on his sales of commodities.

In general, consumer h's situation is more complicated. On the securities market, he has given up $q_h^m(s)$ state-s dollars, but he has received

$$b_h^m(s) \frac{\sum_{k=1}^{k=n} q_k^m(s)}{\sum_{k=1}^{k=n} b_k^m(s)}$$

state-s dollars in payment for his securities-market bid $b_h^m(s)$. Hence, in order to avoid punishment, consumer h must meet the commodity-market budget constraint:

$$\text{(3.2.ii)} \quad \sum_{j=1}^{j=\ell} b_h^j(s) \leq \sum_{j=1}^{j=\ell} \left[q_h^j(s) \frac{\sum_{k=1}^{k=n} b_k^j(s)}{\sum_{k=1}^{k=n} q_k^j(s)} \right] + b_h^m(s) \frac{\sum_{k=1}^{k=n} q_k^m(s)}{\sum_{k=1}^{k=n} b_k^m(s)} - q_h^m(s)$$

for $s = 1,\ldots,r$.

The consumption, $\tilde{x}_h$, of consumer h is given by

$$(3.3)\quad \begin{cases} x_h^i(s) = \omega_h^i - q_h^i(s) + {}_h b^i(s) \dfrac{\sum_{k=1}^{n} q_k^i(s)}{\sum_{k=1}^{n} b_k^i(s)} \\ \qquad\qquad \text{if the Budget Constraints (3.2.i)-(3.2.ii) hold,} \\ x_h^i(s) = 0 \quad \text{otherwise,} \end{cases}$$

for $i = 1,\ldots,\ell$ and $s = 1,\ldots,r$.

The System of Equations (3.3) is consistent with the following auditing-punishment procedure: Trade takes place on the securities market, and if Constraint (3.2.i) is violated, consumer h is punished on the spot market no matter which state of nature occurs, i.e., $x_h(s) = 0$ for $s = 1,\ldots,r$. Then the referee audits the consumers' spot market plans. If it is the case that in some state of nature s, consumer h violates Constraint (3.2.ii), then he is punished on the spot market no matter which state of nature occurs, i.e., $x_h(s') = 0$ for $s' = 1,\ldots,r$.

Some further comments regarding constraints (3.2) are in order. The expression

$$b_h^m(s) \frac{\sum_{k=1}^{n} q_k^m(s)}{\sum_{k=1}^{n} b_k^m(s)} - q_h^m(s)$$

gives the net number of state-s securities purchased by consumer h. The expression $(\sum_k b_k^m(s) \,/\, \sum_k q_k^m(s))$ is the price of the state-s security, so Inequality (3.2.i) says that the value in "general dollars" of net securities purchases must be nonpositive. There are sufficient degrees of freedom to normalize bids on each state's spot market (as long as not all trading posts for that state are closed), which fixes the value of money in that state. Unless all securities trading posts are closed, then we can also normalize the bids on the securities market.

From the Market Game Γ and the random variable s, we have constructed the Securities Game $\tilde{\Gamma}$, which can be thought of as the noncompetitive analogue of the Arrow [1] securities model. It can also be thought of as the noncompetitive analogue of the particular Cass-Shell [6] sunspot model in which there are no restrictions on market participation.[8/] The Securities Game $\tilde{\Gamma}$ is completely specified. The strategy sets are $\tilde{S}_h$ $(h = 1,\ldots,n)$; cf. Equation (3.1). The outcomes $(x_h(1),\ldots,x_h(r))$ $(h = 1,\ldots,n)$ are given by Equation (3.3), and the payoffs are the expected utilities v_h $(h = 1,\ldots,n)$ at probabilities $\{\pi(s)\}_{s=1}^{s=r}$. We adopt the standard definition of Nash Equilibrium.

3.4. Definition. A Nash Equilibrium strategy to the Securities Game $\tilde{\Gamma}$ is a $\tilde{\sigma} \in \tilde{S}$ with the property

$$v_h(\tilde{x}_h(\tilde{\sigma})) = \max_{\tilde{\sigma}'_h \in \tilde{S}_h} \{ v_h(\tilde{x}_h(\tilde{\sigma}|\tilde{\sigma}'_h)) \}$$

for $h = 1,\ldots,n$. The corresponding NE allocation is $\tilde{x}(\tilde{\sigma}) = (\tilde{x}_1(\tilde{\sigma}),\ldots,\tilde{x}_h(\tilde{\sigma}),\ldots,\tilde{x}_n(\tilde{\sigma})) \in \mathbb{R}_{++}^{\ell nr}$.

We next establish that the game $\tilde{\Gamma}$ is "individually rational" for each of the n consumers.

3.5. Lemma. Let $\tilde{\sigma}_h$ be consumer h's best response to the strategies $\tilde{\sigma}_{-h}$ in the Market Game $\tilde{\Gamma}$, i.e.,

$$v_h(\tilde{x}_h(\tilde{\sigma})) = \max_{\tilde{\sigma}'_h \in \tilde{S}_h} \{ v_h(\tilde{x}_h(\tilde{\sigma}|\tilde{\sigma}'_h)) \} .$$

Then, we have

$$v_h(\tilde{x}_h(\tilde{\sigma})) \geq v_h(\tilde{\omega}_h) = v_h(\omega_h, \ldots, \omega_h)$$

$$= \sum_{s=1}^{r} \pi(s) u_h(\omega_h) = u_h(\omega_h) .$$

Proof: If, in response to the strategies, $\tilde{\sigma}_{-h}$, of the others, consumer h plays the trivial strategy $\tilde{\sigma}_h$ given by

$$b_h^i(s) = 0,\ q_h^i(s) = 0,\ b_h^m(s) = 0, \text{ and } q_h^m(s) = 0,$$

for $i = 1,\ldots,\ell$ and $s = 1,\ldots,r$, then from Equation (3.3), we have $x_h^i(s) = \omega_h^i$ for $i = 1,\ldots,\ell$ and $s = 1,\ldots,r$. Hence, we have

$$v_h(\tilde{x}_h(\tilde{\sigma})) \geq v_h(\tilde{\omega}_h) = v_h(\omega_h, \ldots, \omega_h)$$

$$= \sum_{s=1}^{s=r} \pi(s) u_h(\omega_h) = u_h(\omega_h).$$

The inequality above reflects consumer h's ability to "defend his endowments". The last equality above is a consequence of $\sum_{s=1}^{s=r} \pi(s) = 1$. □

We next show that $\tilde{\Gamma}$ has a trivial NE. Later we show that there is also a nontrivial NE for $\tilde{\Gamma}$.

3.6. Lemma. Let $\tilde{\sigma}$ be the vector in $\tilde{S}$ with each component zero. Then $\tilde{\sigma}$ is a NE strategy for $\tilde{\Gamma}$. The corresponding NE allocation is $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_h,\ldots,\tilde{x}_n) = \tilde{\omega} = (\tilde{\omega}_1,\ldots,\tilde{\omega}_h,\ldots,\tilde{\omega}_n) \in \mathbb{R}^{\ell rn}_{++}$.

Proof: Obvious. $\square$

3.7. Definition. A NE strategy is said to be <u>interior</u> if it entails each market (including the markets for securities) being open, i.e., we have $\sum_{k=1}^{k=n} b^j_k(s) > 0$ for $j = 1,\ldots,\ell$ and $s = 1,\ldots,r$, and $\sum_{k=1}^{k=n} b^m_k(s) > 0$ for $s = 1,\ldots,r$.

3.8. Definition. We say that <u>sunspots do not matter</u> if in the allocation of consumption goods, we have

$$x_h(s) = x_h(s') \tag{3.9}$$

for $h = 1,\ldots,n$ and $s,s' = 1,\ldots,r$. Otherwise, <u>sunspots matter</u>. A NE to $\tilde{\Gamma}$ in which Condition (3.9) is satisfied (resp; not satisfied) is called a <u>Nonsunspot NE</u> (resp. <u>Sunspot NE</u>) to $\tilde{\Gamma}$.

It is easy to display an interior Nonsunspot NE to $\tilde{\Gamma}$. This is done in the next proposition.

3.10. Proposition. The Securities Game $\tilde{\Gamma}$ has an interior Nonsunspot NE.

Proof: Let $\sigma = \{(b_k, q_k)\}_{k=1}^{k=n} \in S$ be an interior NE of the certainty game Γ (analyzed in Section 2). From Proposition (2.23), we know that there is such a strategy σ. We now construct $\tilde{\sigma} \in \tilde{S}$ to be an interior Nonsunspot NE to $\tilde{\Gamma}$:

$$
(3.11) \quad
\begin{aligned}
b_h^i(s) &= b_h^i \quad \text{for } i = 1,\ldots,\ell;\ s = 1,\ldots,r;\ h = 1,\ldots,n; \\
q_h^i(s) &= q_h^i \quad \text{for } i = 1,\ldots,\ell;\ s = 1,\ldots,r;\ h = 1,\ldots,n; \\
b_h^m(s) &= \pi(s) \quad \text{for } s = 1,\ldots,r;\ h = 1,\ldots,n; \\
q_h^m(s) &= 1 \quad \text{for } s = 1,\ldots,r;\ h = 1,\ldots,n.
\end{aligned}
$$

No income is being transferred between states if $\tilde{\sigma}$ defined by (3.11) is the strategy vector for $\tilde{\Gamma}$. Hence the Constraint (3.2.i) holds with equality. Then, Constraint (3.2.ii) holds with equality since σ is an interior NE of Γ.

Since all markets are open in the securities game $\tilde{\Gamma}$ for the strategy $\tilde{\sigma}$ described in Equations (3.11), the first-order conditions for utility maximization under binding Constraints (3.2) are necessary and sufficient for optimality. These conditions are

$$
(3.12) \quad \frac{\lambda_h(s)}{\lambda_h(s')} =
$$

$$
\frac{\pi(s)}{\pi(s')} \frac{\partial u_h(x_h(s))/\partial x_h^i(s)}{\partial u_h(x_h(s'))/\partial x_h^j(s')} \frac{\sum_{k\neq h} b_k^i(s)}{\sum_{k\neq h} q_k^i(s)} \left[\frac{\sum_{k=1}^n q_k^i(s)}{\sum_{k=1}^n b_k^i(s)}\right]^2 \frac{\sum_{k\neq h} q_k^j(s')}{\sum_{k\neq h} b_k^j(s')} \left[\frac{\sum_{k=1}^n b_k^j(s')}{\sum_{k=1}^n q_k^j(s')}\right]^2
$$

and

$$(3.13) \qquad \frac{\lambda_h(s)}{\lambda_h(s')} = \frac{\sum_{k\neq h} q_k^m(s)}{\sum_{k\neq h} b_k^m(s)} \left[\frac{\sum_{k=1}^{n} b_k^m(s)}{\sum_{k=1}^{n} q_k^m(s)} \right]^2 \frac{\sum_{k\neq h} b_k^m(s')}{\sum_{k\neq h} q_k^m(s')} \left[\frac{\sum_{k=1}^{n} q_k^m(s')}{\sum_{k=1}^{n} b_k^m(s')} \right]^2$$

for $h = 1,\ldots,n$; $i,j = 1,\ldots,\ell$; and $s,s' = 1,\ldots,r$; and $\lambda_h(s)$ and $\lambda_h(s')$ are (respectively) the Kuhn-Tucker-Langrange multipliers associated with Constraint (3.2.ii) for states s and s'.

Substitute the data from (3.11) into the right-hand side of Equation (3.13), which is consistent if

$$(3.14) \qquad \frac{\lambda_h(s)}{\lambda_h(s')} = \frac{\pi(s)}{\pi(s')} .$$

If we substitute from Equation (3.14), Equation (3.12) must hold because of First-order Condition (2.13).

Thus, $\tilde{\sigma} \in \tilde{S}$ defined by Equations (3.11) is an interior NE for $\tilde{\Gamma}$. Since we have $x_h(s) = x_h(s')$ for $s,s' = 1,\ldots,r$ and $h = 1,\ldots,n$, $\tilde{\sigma}$ is also a Nonsunspot NE for the Securities Game $\tilde{\Gamma}$. $\square$

3.15. Remark. A careful reading of the proof of Proposition (3.10) shows that for every NE strategy $\sigma \in S$ (with corresponding allocation $x(\sigma) \in \mathbb{R}_{++}^{\ell n}$) for the Market Game Γ there is an "equivalent" NE strategy $\tilde{\sigma} \in \tilde{S}$ for the Securities Game $\tilde{\Gamma}$ (with corresponding allocation $\tilde{x}(\tilde{\sigma}) \in \mathbb{R}_{++}^{\ell nr}$). The strategies σ and $\tilde{\sigma}$ are equivalent in the sense

$$(3.16) \qquad x_h(s; \tilde{\sigma}) = x_h(\sigma)$$

for $s = 1,\ldots,r$ and $h = 1,\ldots,n$. Thus $\tilde{x}(\tilde{\sigma})$ is a Nonsunspot NE for $\tilde{\Gamma}$. Essentially, the NE to Γ reappear as the Nonsunspot NE to $\tilde{\Gamma}$.

In the remainder of this section we show that Sunspot NE to $\tilde{\Gamma}$ must exist. We also analyze the relationship of these Sunspot NE to the Correlated Equilibria of the certainty game Γ, and discuss the macroeconomic importance of various types of Sunspot NE.

3.17. Proposition. There is a Sunspot NE $\tilde{\sigma} \in \tilde{S}$ to the Securities Game $\tilde{\Gamma}$ if and only if the endowment vector $\omega \in \mathbb{R}^{\ell n}_{++}$ in the corresponding (certainty) Market Game Γ is not Pareto-optimal.

Proof: Assume that ω is not Pareto-optimal. Then there are at least two NE strategies for Γ, $\sigma' = 0 \in S$ (with the corresponding allocation $x' = ((x_1)',\ldots,(x_h)',\ldots,(x_n)') = \omega \in \mathbb{R}^{\ell n}_{++}$), and an interior NE strategy $\sigma'' = \{(\sigma_h)''\}_{h=1}^{h=n} = \{((b_h)'',(q_h)'')\}_{h=1}^{h=n}$ (with the corresponding allocation $x'' = ((x_1)'',\ldots,(x_h)'',\ldots,(x_n)'') \in \mathbb{R}^{\ell n}_{++}$). Cf. Lemma (2.5) and Proposition (2.23). From Proposition (2.20), we know that x' and x'' are not equal.

Partition the states of nature $\{1,\ldots,r\}$ into two subsets, A (for "autarky") and I (for "interior"). We have $A \cup I = \{1,\ldots,r\}$, $A \cap I = \emptyset$, $A \neq \emptyset$, and $I \neq \emptyset$. We construct the Sunspot NE $\tilde{\sigma}$ for the Securities Game $\tilde{\Gamma}$ from these two NE of the Market Game Γ as follows:

$$(3.18.A) \quad \begin{cases} b_h^i(s) = 0, \\ q_h^i(s) = 0, \\ b_h^m(s) = 0, \\ q_h^m(s) = 0, \end{cases}$$

for $s \in A$, $h = 1,\ldots,n$, and $i = 1,\ldots,\ell$; and

$$
(3.18.\mathrm{I}) \quad \begin{cases} b_h^i(s) = (b_h^i)^{*}, \\ q_h^i(s) = (q_h^i)^{*}, \\ b_h^m(s) = 0, \\ q_h^m(s) = 0, \end{cases}
$$

for $s \in I$, $h = 1,\ldots,n$, and $i = 1,\ldots,\ell$, where $(b_h^i)^{*}$ and $(q_h^i)^{*}$ are bids and offers in the interior NE strategy σ^{*} for the Market Game Γ. The strategy $\tilde{\sigma} \in \tilde{S}$ is clearly a Sunspot NE for the Securities Game $\tilde{\Gamma}$.

Assume that ω is Pareto-optimal in Γ. Clearly, then $\tilde{\omega}$ is also Pareto-optimal in $\tilde{\Gamma}$. Assume that $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_h,\ldots,\tilde{x}_n)$ is a Sunspot NE allocation in $\tilde{\Gamma}$. By Lemma (3.5), we have

$$
(3.19) \qquad v_h(\tilde{x}_h) \geq v_h(\omega_h,\ldots,\omega_h) = u_h(\omega_h).
$$

Because of the strict concavity of u_h a Sunspot allocation cannot be Pareto-optimal. We have a contradiction. If $\tilde{\omega}$ is Pareto-optimal in $\tilde{\Gamma}$, there are no Sunspot NE for $\tilde{\Gamma}$. Indeed, the only NE allocation is $\tilde{x} = \tilde{\omega} = (\tilde{\omega}_1,\ldots,\tilde{\omega}_n) \in \mathbb{R}_{++}^{\ell nr}$. (The vector $\tilde{x}$ is both an interior NE allocation for $\tilde{\Gamma}$ <u>and</u> an autarkic NE allocation for $\tilde{\Gamma}$.)

The proof of Proposition (3.17) is complete. $\square$

<u>**3.20. Remark.**</u> The Sunspot NE allocation $\tilde{x}$ constructed in Proposition (3.17) is a lottery over (certainty) NE from the underlying Market Game Γ. As such, the allocation $\tilde{x}$ is also a Correlated Equilibrium allocation to the Market Game Γ (cf. Aumann [2,3]), where the probability mechanism is based on the random variable s and probabilities $\pi(s)$, used in defining the securities

game $\tilde{\Gamma}$. (Note that this particular probability mechanism is degenerate in that each consumer's signal, sunspot activity, is perfectly correlated with that of each other consumer. Aumann [2,3], of course, allows for — indeed, he makes much of -- imperfectly correlated signals.[9/])

The Sunspot NE to the Securities Game $\tilde{\Gamma}$ in Proposition (3.17) is constructed as a lottery over (i) the closed-commodity-market NE to Γ and (ii) the thick-commodity-market NE to Γ in which each consumer offers all of his endowments. There are indeed a continuum of NE to Γ, parameterized by market thickness; cf., in general, our Proposition (2.16) and, for the two-person case, Shapley [15]. Next we consider an example of a 2×2 Market Game Γ. We calculate three interior NE to this game and show how the equilibria relate to market thickness. Later, we use these calculated equilibria in constructing examples of interior and noninterior Sunspot NE to the related Securities Game $\tilde{\Gamma}$.

3.21. Example. Let there be two consumers $(h = 1,2)$ and two commodities $(i = 1,2)$, so that $n = 2$ and $\ell = 2$. The following data about consumer preferences and endowments complete the description of the (certainty) Market Game Γ:

$$(3.22)\quad \left\{\begin{array}{l} u_h(x_h^1,x_h^2) = \log x_h^1 + \log x_h^2 \quad \text{for} \quad h = 1,2 \\ \\ \text{and} \\ \\ \omega_1 = (\omega_1^1,\omega_1^2) = (80,20);\ \omega_2 = (\omega_2^1,\omega_2^2) = (20,80). \end{array}\right.$$

Solution 1 to Example (3.21): The example exhibits a skew-symmetry between the two consumers. Hence, if each of the consumers offers 100% of his endowment for sale, we have the skew-symmetric interior NE to Γ displayed below. This is a thick-market solution. Trading is substantial, but since this game is neither cooperative nor perfectly competitive, the allocation of consumption goods is still far from Pareto-optimal.

	Commodity 1	Commodity 2
b_1	.3333	.1667
q_1	80.0000	20.0000
x_1	66.6667	33.3333
b_2	.1667	.3333
q_2	20.0000	80.0000
x_2	33.3333	66.6667

Solution 1 to the 2×2 Game Γ of Example (3.21): Each Consumer Offers All of his Endowments

Solution 2 to Example (3.21): Here markets are thinner than before. Each consumer offers for sale only 25% of his endowments. Trading is substantially less than in the first example. Lack of consumer confidence is self-justifying. Skew-symmetry is preserved. Each consumer is worse off in Solution 2 than in Solution 1. The NE is interior.

	Commodity 1	Commodity 2
b_1	.2892	.2108
q_1	20.0000	5.0000
x_1	74.4603	25.5397
b_2	.2108	.2892
q_2	5.0000	20.0000
x_2	25.5397	74.4603

Solution 2 to the 2×2 Game Γ of Example (3.21): Each Consumer Offers 25% of his Endowments

Solution 3 to Example (3.21): The NE is interior, but the skew-symmetry is broken. Consumer 1 offers 100% of his endowments, but Consumer 2 offers only 25%. These strategies are self-justifying. Markets are thin relative to those in Solution 1. Each consumer is worse off than in Solution 1. Consumer 2 is worse off here than he is in Solution 2, while Consumer 1 is better off here than he is in Solution 2. Indeed, in moving from Solution 2 to Solution 3, Consumer 1 gives up .2367 units of Commodity 1, the marginal utility of which is relatively low, in exchange for 4.2573 units of Commodity 2, the marginal utility of which is relatively high.

	Commodity 1	Commodity 2
b_1	.6836	.1618
q_1	80.0000	20.0000
x_1	74.2236	29.7970
b_2	.0992	.0554
q_2	5.0000	20.0000
x_2	25.7764	70.2029

Solution 3 to the 2×2 Game Γ of Example (3.21): Consumer 1 Offers All of his Endowments; Consumer 2 Offers 25% of his Endowments

The following is an example of a Securities Game $\tilde{\Gamma}$ which is based on the Market Game Γ described by Equations (3.22) (cf. Example (3.21)).

3.23. Example. Let Γ be described by the Data (3.22). Let there be two states $s = \alpha,\beta$ (i.e., $r = 2$) and assume that the extrinsic random variable s obeys the probability law $\pi(\alpha) = \pi(\beta) = 1/2$. Let $\tilde{\Gamma}$ be the corresponding securities game.

Next, we compute three Sunspot NE for $\tilde{\Gamma}$. The allocation of resources varies across states of nature as market thickness varies. These solutions establish:

(3.24)
- (i) Some Sunspot NE to $\tilde{\Gamma}$ are interior; others are not.
- (ii) Some interior Sunspot NE to $\tilde{\Gamma}$ involve nonzero net trades on the securities market; others do not.
- (iii) Some interior Sunspot NE allocations to $\tilde{\Gamma}$ are also Correlated Equilibrium allocations to Γ; others are not.

<u>**Solution 1 to Example (3.23)**</u>: State α is the "good state", which corresponds to the interior NE for Γ given in Solution 1 to (3.21). State β is the "bad state", which corresponds to the interior NE for Γ given in Solution 2 to (3.21). "Confidence" drops in moving from α to β; all offers are reduced by 75%, creating thin markets. Each consumer is worse off in β than in α. The price of the α- security in terms of the β- security is

$$\frac{(155.8731 + 155.8731) \;/\; (100 + 100)}{(44.1269 + 44.1269) \;/\; (100 + 100)} = 3.53 \;.$$

Despite their relative poverty in state β, each consumer is (just) willing to give up 3.53 state-β dollars in exchange for a single state-α dollar. "Needs" are greater in β than in α, but "opportunities" in α are very much greater than in β, which is reflected in the exchange rate between α-dollars and β-dollars.

Because of the various symmetries, it turns out that in equilibrium Consumer 1 and Consumer 2 have the same relative utility weights for state-α income versus state-β income. There are then no social gains to be made from transferring income across the states of nature. Hence, we have an interior Sunspot NE to $\tilde{\Gamma}$ in which net securities trades are zero. Therefore, this <u>interior</u> Sunspot NE allocation can be taken as a lottery over interior NE for the (certainty) Market Game Γ; (cf. Solutions 1 and 2 to Example (3.21)). Furthermore, this interior Sunspot NE allocation (to $\tilde{\Gamma}$) is also a Correlated Equilibrium allocation for the (certainty) Market Game Γ.

	State α			State β		
	Comm. 1	Comm. 2	Security	Comm. 1	Comm. 2	Security
$\tilde{b}_1$	.3333	.1667	155.8731	.2892	.2108	44.1269
$\tilde{q}_1$	80.0000	20.0000	100.0000	20.0000	5.0000	100.0000
$\tilde{x}_1$	66.6667	33.3333	- -	74.4603	25.5397	- -
$\tilde{b}_2$	.1667	.3333	155.8731	.2108	.2892	44.1269
$\tilde{q}_2$	20.0000	80.0000	100.0000	5.0000	20.0000	100.0000
$\tilde{x}_2$	33.3333	66.6667	- -	25.5397	74.4603	- -

Solution 1 to the Game $\tilde{\Gamma}$ defined in Example (3.23): Markets are Thick in State α and Thin in State β. Net Securities Purchases are Zero.

Solution 2 to Example (3.23): State α is the good state. Each consumer offers 100% of his endowment in state α; the commodity markets in state α are thick. Consumer 1 offers 100% of his endowment in state β, the bad state, but Consumer 2 offers only 25% of his endowment in state β. Both consumers are worse off in β than in α, although the impact on Consumer 2 turns out, in this particular case, to be the more dramatic. The securities markets are closed. Hence, this solution can be taken as a lottery over interior NE Solutions (1) and (3) (to Examples (3.21)) in the (certainty) Market Game Γ. This solution is a Sunspot NE which is not interior. The corresponding Sunspot NE allocation is also a Correlated Equilibrium allocation to the (certainty) Market Game Γ.

	State α			State β		
	Comm. 1	Comm. 2	Security	Comm. 1	Comm. 2	Security
$\tilde{b}_1$	.3333	.1667	0.	.6836	.1618	0.
$\tilde{q}_1$	80.0000	20.0000	0.	80.0000	20.0000	0.
$\tilde{x}_1$	66.6667	33.3333	- -	74.2236	29.7970	- -
$\tilde{b}_2$	.1667	.3333	0.	.0992	.0554	0.
$\tilde{q}_2$	20.0000	80.0000	0.	5.0000	20.0000	0.
$\tilde{x}_2$	33.3333	66.6667	- -	25.7764	70.2029	- -

Solution 2 to the Game $\tilde{\Gamma}$ defined in Example (3.23): Consumer 2 Reduces his Offers to 25% in State β. The Securities Markets are Closed.

Solution 3 to Example (3.23): Solution 3 is very much like Solution 2 with one very important difference: In Solution 3, the securities markets are open and net securities purchases are nonzero. The price of the α-security in terms of the β-security is

$$\frac{(116.3231 + 116.3031) \ / \ (100 + 100)}{(83.6768 + 83.6968) \ / \ (100 + 100)} = 1.39$$

Consumer 1's purchases of the α-security (or better, α-money) are $200[(116.3211) \ / \ (116.3211 + 116.3031)] = 100.0077$ units. His <u>net</u> purchases of the α-money are hence .0077 units. Consumer 1's bids for commodities in state α sum to .5087 (= .3376 + .1710) state-α dollars, of which .0077 state-α dollars (amounting to 1.5% of the total) are financed by his purchases of α-money in the securities market.

Consumer 1 transfers income into state α, while consumer 2 transfers income into state β. Consumer 1 seeks commodity 2 in state β, but Consumer 2 offers little of this commodity. Hence, Consumer 1 parts with commodity 1 in state β in exchange for state-β money, a substantial portion of which he then exchanges for state-α money. The state-α money is used to finance his purchases of commodities in state α.

Compare Solutions (2) and (3) (Example (3.23)). With open security markets (Solution 3), Consumer 1 increases his consumption of both commodities in (the good state) α and reduces his consumption of both commodities in (the bad state) β.

Solution 3 is especially noteworthy. It is an interior Sunspot NE solution to $\tilde{\Gamma}$ with open and <u>active securities markets</u>. Hence, this solution <u>cannot</u> be considered to be a lottery over NE solutions to Γ. The corresponding Sunspot NE allocation is <u>not</u> a Correlated Equilibrium allocation to the (certainty) Market Game Γ. (Every Correlated Equilibrium allocation to Γ is also a Sunspot Equilibrium allocation to $\tilde{\Gamma}$, but some Sunspot Equilibrium allocations to $\tilde{\Gamma}$ are not Correlated Equilibrium allocations to Γ.)

	State α			State β		
	Comm. 1	Comm. 2	Security	Comm. 1	Comm. 2	Security
$\tilde{b}_1$	.3376	.1710	116.3231	.6741	.1586	83.6768
$\tilde{q}_1$	80.0000	20.0000	100.0000	80.0000	20.0000	100.0000
$\tilde{x}_1$	67.5204	34.1981	- -	73.3364	29.0214	- -
$\tilde{b}_2$	.1624	.3290	116.3031	.1072	.0600	83.6968
$\tilde{q}_2$	20.0000	80.0000	100.0000	5.0000	20.0000	100.0000
$\tilde{x}_2$	32.4796	65.8019	- -	26.6636	70.9786	- -

Solution 3 to the Game $\tilde{\Gamma}$ defined in Example (3.23): Consumer 2 Reduces his Offers to 25% in State β. The Securities Markets are Open, and Net Purchases of Securities are Nonzero.

4. The Large Economy.

Our analysis of the finite Securities Game is completed. Excepting the special case in which endowments are Pareto-optimal, the following is true in the finite Securities Game:

(i) Consumers possess oligopoly and oligopsony power in the markets; as a consequence, outcomes are not Pareto-optimal.

(ii) There is a wide range of rational (i.e., self-justifying) beliefs about market thickness. As a consequence, there are many rational-expectations equilibria.

(iii) There are rational-expectations equilibria in which sunspots do not matter. There are rational-expectations equilibria in which sunspots matter.

In this section, we analyze the replication economy — based on the Securities Game $\tilde{\Gamma}$ — in which the number of consumers is arbitrarily large.

Excepting the special case in which endowments are Pareto-optimal, the following is true in the limiting Securities Game:

(i) Consumers possess neither oligopoly nor oligopsony power in the markets. The economy is "competitive," but since some markets may be closed[10], the allocation of consumption goods is not necessarily Pareto-optimal.

(ii) There are many rational (i.e., self-justifying) beliefs about which markets are open and which are closed; in fact any prespecified set of markets can be closed. As a consequence, there are many rational-expectations equilibria.

(iii) There are rational-expectations equilibria in which sunspots do not matter (some, but not all, of which are Pareto-optimal). There are also rational-expectations equilibria in which sunspots matter (all of which are not Pareto-optimal).

We now give a brief description of the procedure for replicating $\tilde{\Gamma}$. (For a more complete description of the replication technique, see [16]).

Let $\tilde{\Gamma}^v$ be the v-th replication of $\tilde{\Gamma}$. $\tilde{\Gamma}^v$ is thus the Securities Game in which there are v $(v = 1,2,\ldots)$ consumers exactly like consumer h $(h = 1,\ldots,n)$ from the original Securities Game $\tilde{\Gamma}$. Hence, we say that there are v consumers of type h $(h = 1,\ldots,n)$ in $\tilde{\Gamma}^v$. (Notice that $\tilde{\Gamma}^1 = \tilde{\Gamma}$.) A strategy vector $\tilde{\sigma}^v$ for the game $\tilde{\Gamma}^v$ is given by

$$\tilde{\sigma}^v = \{\{\tilde{\sigma}_h^t\}_{t=1}^{t=v}\}_{h=1}^{h=n} = \{\{(\tilde{b}_h^t, \tilde{q}_h^t)\}_{t=1}^{t=v}\}_{h=1}^{h=n} \in \mathbb{R}_+^{2r(\ell+1)vn}.$$

In what follows, we focus on symmetric strategy vectors, i.e., vectors $\tilde{\sigma}^v$

with the property

$$\tilde{\sigma}_h^t = \tilde{\sigma}_h^{t'}$$

for $t,t' = 1,\ldots,v$ and $h = 1,\ldots,n$. Generally, we shall find it convenient to describe a symmetric <u>strategy vector</u> $\tilde{\sigma}^v$ in the smaller space $\mathbb{R}_+^{2r(\ell+1)n}$, namely

$$\tilde{\sigma}^v = \{\tilde{\sigma}_h^v\}_{h=1}^{h=n} = \{(\tilde{b}_h^v,\tilde{q}_h^v)\}_{h=1}^{h=n} \in \mathbb{R}_+^{2r(\ell+1)n}.$$

This should not cause confusion.

Let $\{v\}$ denote the sequence $(1,\ldots,v,\ldots)$, $\{\tilde{\Gamma}^v\}$ the sequence $(\tilde{\Gamma}^1,\ldots,\tilde{\Gamma}^v,\ldots)$, and $\{\tilde{\sigma}^v\}$ the sequence of symmetric strategies $(\tilde{\sigma}^1,\ldots,\tilde{\sigma}^v,\ldots)$. We know there is an increasing subsequence $\{\nu\}$ of $\{v\}$ with the property that on this subsequence the symmetric strategies $\tilde{\sigma}^\nu$ tend to a limiting strategy, say, $\tilde{\sigma}$. Allowing ourselves some carelessness of notation (since we are imprecise about the subsequence), we describe this limiting process as $\tilde{\sigma}^\nu \to \tilde{\sigma} \in \mathbb{R}_+^{2r(\ell+1)n}$ as $\nu \to \infty$.

We consider next the well-behaved case of interior NE in the replicated Securities Game, $\tilde{\Gamma}^v$. From Proposition (3.10), we know that there is an interior NE strategy $\tilde{\sigma}^v$ to $\tilde{\Gamma}^v$ for $v = 1,2,\ldots$. We make the further claim that there is a symmetric, interior NE strategy $\tilde{\sigma}^v$ to $\tilde{\Gamma}^v$ for $v = 1,2,\ldots$. (The proof of this claim[11/] requires one modification in our proof of Proposition (2.23). Consider the mapping which takes the point b to the point b'. Replace the unit simplex, here $\Delta^{r(\ell+1)vn}$, with the subset of <u>symmetric</u> bids from $\Delta^{r(\ell+1)vn}$.)

4.1. Proposition. Let $\tilde{\sigma}^{\nu} = (\tilde{\sigma}_1^{\nu},\ldots,\tilde{\sigma}_h^{\nu},\ldots,\tilde{\sigma}_n^{\nu}) \in \mathbb{R}_+^{2r(\ell+1)n}$ be a symmetric, interior NE strategy for the replicated Securities Game $\tilde{\Gamma}^{\nu}$, with the property

$$\sum_{k=1}^{k=n} (b_k^i(s))^{\nu} > \varepsilon , \quad \sum_{k=1}^{k=n} (q_k^i(s))^{\nu} > \varepsilon ,$$

$$\sum_{k=1}^{n} (b_k^m(s))^{\nu} > \varepsilon , \quad \sum_{k=1}^{k=n} (q_k^m(s))^{\nu} > \varepsilon ,$$

for each ν; $i = 1,\ldots,\ell$; $s = 1,\ldots,r$ and some $\varepsilon > 0$ (and independent of ν). Let $\tilde{\sigma} \in \mathbb{R}_+^{2r(\ell+1)n}$ be the limit of the strategy vectors (after many replications).[12]

Then the corresponding allocation vector $\tilde{x}(\tilde{\sigma}) \in \mathbb{R}_{++}^{\ell rn}$ is also a competitive equilibrium allocation vector for the economy $\tilde{\Gamma}$. (Of course, $\tilde{x}(\tilde{\sigma})$ is then also a competitive equilibrium allocation for $\tilde{\Gamma}^{\nu}$ for $\nu = 1,2,\ldots,$.)

Proof: Consider $\tilde{\sigma}^{\nu} \in \mathbb{R}_+^{2r(\ell+1)n}$, a symmetric, interior NE strategy vector for the game $\tilde{\Gamma}^{\nu}$. Consider one consumer of type h. Fix the bids and offers of all other consumers at their NE amounts. Let $z_h^i(s)$ be the bid for commodity i in state s by this single consumer and let $y_h^i(s)$ be the corresponding offer. Let $z_h^m(s)$ and $y_h^m(s)$ be (respectively) his bid for and offer of state-s money. Therefore $\tilde{\sigma}_h^{\nu} = (\tilde{b}_h^{\nu},\tilde{q}_h^{\nu})$ is the nonnegative value of $(z_h^1(1),\ldots,z_h^{\ell}(r),z_h^m(1),\ldots,z_h^m(r),y_h^1(1),\ldots,y_h^{\ell}(r),y_h^m(1),\ldots,y_h^m(r))$ which maximizes $u_h(\tilde{x}_h^{\nu})$ subject to

$$(x_h^i(s))^\nu = \frac{z_h^i(s)[y_h^i(s) + (\nu-1)(q_h^i(s))^\nu + \nu \sum_{k\neq h} (q_k^i(s))^\nu]}{z_h^i(s) + (\nu-1)(b_h^i(s))^\nu + \nu \sum_{k\neq h} (b_k^i(s))^\nu}$$

(4.2)

$$= \frac{z_h^i(s)[(1/\nu)y_h^i(s) + ((\nu-1)/\nu)(q_h^i(s))^\nu + \sum_{k\neq h} (q_k^i(s))^\nu]}{(1/\nu)(z_h^i(s)) + ((\nu-1)/\nu)(b_h^i(s))^\nu + \sum_{k\neq h} (b_k^i(s))^\nu} ,$$

for $i = 1,\ldots,\ell$; $s = 1,\ldots,r$; $h = 1,\ldots,n$,

(4.3)

$$\sum_{j=1}^{\ell} z_h^j(s) \leq \sum_{j=1}^{\ell} \left[\frac{y_h^i(s)[\sum_{k\neq h} (b_k^j(s))^\nu + ((\nu-1)/\nu)(b_h^j(s))^\nu + (1/\nu)\, z_h^j(s)]}{\sum_{k\neq h} (q_k^j(s))^\nu + ((\nu-1)/\nu)(q_h^j(s))^\nu + (1/\nu)\, y_h^j(s)} \right] + z_h^m(s) \frac{\sum_{k\neq h} (q_k^m(s))^\nu + ((\nu-1)/\nu)(q_h^m(s))^\nu + (1/\nu)y_h^m(s)}{\sum_{k\neq h} (b_k^m(s))^\nu + ((\nu-1)/\nu)(b_h^m(s))^\nu + (1/\nu)\, z_h^m(s)} - y_h^m(s)$$

for $s = 1,\ldots,r$, and

$$\text{(4.4)} \quad \sum_{s=1}^{r} \left[z_h^m(s) - y_h^m(s) \frac{\sum_{k\neq h} (b_k^m(s))^\nu + ((\nu-1)/\nu)(b_h^m(s))^\nu + (1/\nu)\, y_h^m(s)}{\sum_{k\neq h} (q_k^m(s))^\nu + ((\nu-1)/\nu)(q_h^m(s))^\nu + (1/\nu)\, z_h^m(s)} \right] \leq 0 ,$$

where $\tilde{x}_h^\nu$ is $((x_h^1(1))^\nu,\ldots,(x_h^i(s))^\nu,\ldots,(x_h^\ell(r))^\nu)$.

Analysis of the System (4.2)-(4.4) establishes that the allocation vector $(\tilde{x})^{\nu}$ is a continuous function of the strategy vector $(\tilde{\sigma})^{\nu}$ and the inverse of the "replication number" ν. We know that $((1/\nu), \tilde{\sigma}^{\nu}) \to (0, \boldsymbol{\tilde{\sigma}})$ and hence because of this continuity, we have

$$\tilde{x}(\tilde{\sigma}^{\nu}) \to \tilde{x}(\boldsymbol{\tilde{\sigma}}).$$

That is, $\boldsymbol{\tilde{\sigma}}$ is the utility-maximizing solution to the limiting maximization problem. Let $\boldsymbol{\tilde{x}} = \tilde{x}(\boldsymbol{\tilde{\sigma}})$.

Introduce the following notation (where boldface is used to indicate the components of the limit strategy $\boldsymbol{\tilde{\sigma}}$):

$$p^{i}(s) = \frac{\sum_{k=1}^{n} \mathbf{b}_{k}^{i}(s)}{\sum_{k=1}^{n} \mathbf{q}_{k}^{i}(s)} , \quad p^{m}(s) = \frac{\sum_{k=1}^{n} \mathbf{b}_{k}^{m}(s)}{\sum_{k=1}^{n} \mathbf{q}_{k}^{m}(s)} ,$$

(4.5)

$$\text{and} \quad x_{h}^{m}(s) = \frac{\mathbf{b}_{h}^{m}(s)}{p^{m}(s)} - \mathbf{q}_{h}^{m}(s) .$$

Take the limit $\nu \to \infty$ and substitute from Equations (4.5) in the System (4.2)-(4.4) to yield:

$$(4.6) \qquad x_{h}^{i}(s) = z_{h}^{i}(s)/p^{i}(s) ,$$

$$(4.7) \qquad \sum_{j=1}^{\ell} z_{h}^{j}(s) \leq \sum_{j=1}^{\ell} p^{j}(s)\, \omega_{h}^{j} + x_{h}^{m}(s) ,$$

$$(4.8) \qquad \sum_{s=1}^{r} p^{m}(s)\, x_{h}^{m}(s) \leq 0 .$$

Substituting Equation (4.6) into Inequality (4.7) and recopying Inequality (4.8), yields the following system:

$$\sum_{j=1}^{\ell} p^j(s)\; x_h^j(s) \;\leq\; \sum_{j=1}^{\ell} p^j(s)\; \omega_h^j \;+\; x_h^m(s)$$

(4.9) for $s = 1,\ldots,r$ and

$$\sum_{s=1}^{r} p^m(s)\; x_h^m(s) \;\leq\; 0$$

for $h = 1,\ldots,n$. We recognize the System (4.9) as the constraints faced by consumer h in an (Arrow) securities-market competitive equilibrium (which is indeed a competitive equilibrium for the economy $\tilde{\Gamma}$). Here $p^j(s)$ is interpreted as the competitive price of commodity j on the state - s (spot) commodity market, $p^m(s)$ is the price of security s (or the price of state-s dollars), and $x_h^m(s)$ is consumer h's purchases of state-s dollars.

Since $\varepsilon > 0$ is independent of ν, we have that the prices are well defined and satisfy the inequalities

$$p^j(s) > 0 \quad \text{for} \quad j = 1,\ldots,\ell \quad \text{and} \quad s = 1,\ldots,r,$$

$$p^m(s) > 0 \quad \text{for} \quad s = 1,\ldots,r.$$

It only remains to show that in the limit, all materials balance. Since we have

$$\sum_{k=1}^{n} (x_k^j(s))^{\nu} = \sum_{k=1}^{n} (\omega_k^j)$$

for $j = 1,\ldots,\ell$, $s = 1,\ldots,r$ and each ν it follows that

$$\sum_{k=1}^{n} x_k^j(s) = \sum_{k=1}^{n} \omega_k^j$$

for $j = 1,\ldots,\ell$ and $s = 1,\ldots,r$. Using Definitions (4.5) yields

$$\sum_{k=1}^{k=n} x_k^m(s) = \frac{\sum_{k=1}^{n} b_k^m(s)}{p^m(s)} - \sum_{k=1}^{n} q_k^m(s) = 0.$$

We have shown that $\tilde{\sigma}^\nu \to \tilde{\sigma}$ and that $\tilde{x}^\nu \to \tilde{x}$ where $\tilde{x}$ is a competitive-equilibrium allocation for the economy $\tilde{\Gamma}$. Because $\varepsilon > 0$ is independent of ν, we have that in this competitive equilibrium all markets are open. $\square$

We turn to the analysis of the limiting economy in which some markets are closed. Remember, commodity market i in state s is said to be closed if $\sum_{k=1}^{k=n} b_k^i(s) = 0$, which implies $\sum_{k=1}^{k=n} q_k^i(s) = 0$. Security market s is said to be closed if $\sum_{k=1}^{k=n} b_k^m(s) = 0$, which is consistent with $\sum_{k=1}^{k=n} q_k^m(s) = 0$. There are $(\ell+1)r$ trading posts if we count the commodity markets separately for each state. Any subset of these $(\ell+1)r$ markets can be prespecified to be closed and there is a NE to $\tilde{\Gamma}^\nu$ consistent with their closure. Let $C(s)$ be the set of closed commodity markets in state-s. Let $C(m)$ be the set of closed security markets. Then, if C is the set of closed markets, we have $C = C(1) \cup \ldots \cup C(s) \cup \ldots C(r) \cup C(m)$.

4.10. Proposition. Let $\tilde{\sigma}^\nu \in \mathbb{R}_+^{2r(\ell+1)n}$ be a symmetric NE strategy for the replicated Securities Game $\tilde{\Gamma}^\nu$, with the property that some prespecified set of markets is closed. Let $\tilde{\sigma} \in \mathbb{R}_+^{2r(\ell+1)n}$ be the limit of the strategy vectors $\tilde{\sigma}^\nu$.

Then, the corresponding allocation vector $\tilde{x}(\tilde{\sigma})$ is also a competitive-equilibrium allocation vector for the economy $\tilde{\Gamma}$ but with some markets (including the prespecified markets) closed.

Furthermore, for each $\tilde{\Gamma}$, there exist $\{\tilde{\sigma}^{\nu}\}$ with the property that in the limit the prespecified markets are closed but all other markets are open.

Proof: The proof is parallel to the one given for Proposition (4.1). It is obvious that there exist NE $\tilde{\sigma}^{\nu}$ with the prespecified set of markets C closed. A simple extension of Proposition (2.23) yields that for each ν there is a NE with the properties that markets in C are closed and all other markets are open. An extension of the proof of (2.23) (see also Peck and Shell [13]), yields these markets can be uniformly open with offers and bids bounded above some $\varepsilon > 0$ (independent of ν). $\square$

There is always some limit allocation which is a Nonsunspot Competitive-Equilibrium allocation for the economy $\tilde{\Gamma}$. There is nearly always some limit allocation which is a Sunspot Competitive Equilibrium allocation for the economy defined by $\tilde{\Gamma}$, but in which some markets are closed. This is shown in the next proposition.

4.11. Proposition. (i) For each Securities Game $\tilde{\Gamma}$, there is always a limiting allocation $\tilde{x}(\tilde{\sigma})$ which is a Nonsunspot Competitive Equilibrium allocation. (ii) For each Securities Game $\tilde{\Gamma}$ in which the initial allocation of resources is not Pareto-optimal, there is a limiting allocation $\tilde{x}(\tilde{\sigma})$ which is a Sunspot allocation and is competitive with respect to the set of open markets.

Proof: Part (i) follows from Proposition (4.1) and the fact that with all markets open a competitive allocation must be a (Pareto-optimal) Nonsunspot allocation.

Part (ii). Prespecify that all securities markets are closed and that in some state s' all commodity markets are closed. It follows from Proposition (2.23), that there is a NE to $\tilde{\Gamma}$ satisfying the above specification and also having the property that all commodity markets are open in some state $s \neq s'$. By Proposition (2.20), we know there is some h with the property $(x_h(s))^\nu \neq (x_h(s'))^\nu$ for each ν, since ω is not Pareto-optimal. Choose offers in state s to be independent of ν (e.g., $(q_h(s))^\nu = \omega_h$ for each ν) so that in the limit, markets in state s remain open and hence in the limit $x_h(s) \neq x_h(s')$. Using Proposition (4.10) completes the proof of Part (ii).[13] $\square$

4.12. Remark. The persistence of Sunspot Equilibria as the economy becomes large might come as a surprise to readers of the Cass-Shell paper [6]: Here market participation is unrestricted; in particular every consumer is "born" before sunspots are observed. The persistence of Sunspot Equilibria should come as less of a surprise after reading the Mas-Colell paper [10], which provides a description of competitive equilibrium in which some markets are closed. If we consider "competition" to be a limit of NE as the economy becomes large, then for the competitive economy it is not necessary to seek an explanation of why a market is closed. A closed market is closed because economic actors believe it to be closed.

Notes

1. Of course, macro-economists are quite familiar with the general notion of "market uncertainty". Indeed, Keynes argued in The General Theory that "animal spirits" (or "market psychology" or "market uncertainty") play a central role in the determination of national income.

2. We are currently studying the (stabilizing and destabilizing) effects of money taxes on imperfectly competitive economies. The version of the market game adopted here seems to be especially well suited for the taxation study.

3. This specification of the budget constraint places no restrictions on credit, closely following that of Postlewaite and Schmeidler [14].

4. This bankruptcy rule is somewhat arbitrary. However, since in equilibrium no consumer actually goes bankrupt, the rule allows us to avoid the more complicated bankruptcy issues. The game is well-defined, but we offer the following interpretation: The referee punishes the bankrupt consumer by seizing his remaining endowments and his proceeds from trading, but the referee "guarantees" the "checks" written by each consumer including the bankrupts. No consumer (bankrupt or not) offers more than his endowment of physical commodities, so supply obligations are always met.

5. The Lagrangian expression (2.12) does not include multipliers for the constraints $q_h^j \le \omega_h^j$. While these constraints can be binding, the associated multipliers must always be zero. Reducing bids and offers in the proportions given by the prevailing price ratios brings offers below endowments without affecting any of the other constraints, while also maintaining the original consumption and utility levels.

6. In Dubey and Shubik [8], existence of an interior NE is established for a variant of this model which has commodity money and liquidity constraints. Their method of proof does not seem to generalize to our economy. The proof in Pazner-Schmeidler [12] might anticipate ours, but so far we have been unable to locate a copy of their proof.

7. As on the commodity trading posts, when there are no supplies on a security trading post, all bids are lost. When there are no bids for a security, supplies of that security are lost. Hence, in Equation (3.3), we maintain the convention that 0/0 is 0.

8. In the terminology of Cass and Shell [6], this is the case in which $G^I = \emptyset$. If G^I is not empty, then the model becomes more complicated, but Sunspot Equilibrium allocations are even easier to find. In future work on the limit of NE as the economy becomes large, we shall allow for $G^I \neq \emptyset$, i.e., allow for restricted market participation. We shall also allow for nontrivial fiscal policies.

9. Maskin and Tirole [11] have constructed an ingenious example in the competitive setting of a Sunspot Equilibrium based on imperfectly correlated signals. (See also [4], in which the Maskin-Tirole example is

extended to noncompetitive settings.) We are currently engaged in extending the analysis of the present paper to fully incorporate imperfectly correlated randomness.

10. For a definition of "competition with some closed markets," see Mas-Colell [10, p. 191].

11. This method of establishing the existence of a type-symmetric NE was used in Dubey and Shubik [7].

12. For the bids of consumer h in the unreplicated economy to correspond to the bids of a consumer of type h in the replicated economy, we must choose the appropriate normalization. If we have $\sum_{k=1}^{n} \sum_{j=1}^{\ell} b_k^j(s) = 1$, then we must also have $\sum_{k=1}^{n} \sum_{j=1}^{\ell} (b_k^j(s))^{\nu} = 1$ for each ν.

13. Here we construct a Sunspot NE of the limiting Securities Game by closing all markets in one state. In fact, it will usually be the case that a Sunspot Equilibrium can be found by closing any subset of markets in state s (leaving the other markets open) and by closing any other subset of markets in state s' (leaving the other markets open).

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