IV

A Model of Inventive Activity and Capital Accumulation[1]

KARL SHELL
Massachusetts Institute of Technology

1. Introduction

In the contemporary revival of the theory of economic growth, the implications of capital accumulation and population growth have been investigated by several authors, for example, [17,18]. However, technical change has been introduced into these models by means of a continuous secular shift in the aggregate production function—the rate and nature of which are exogenous to the policy variables of the model.

In two models, however, the rate of technical change is related to economic variables. The first is a model introduced by Kaldor in a series of papers [8, 9, 10]. Kaldor posits a positive relation (the technical progress function) between relative changes in per capita productivity and relative changes in gross investment. The technical progress function is an eclectic amalgam of basic technical and institutional forces in a free-enterprise economy. Kaldor takes the Schumpeterian view that the creation of new ideas largely occurs at an autonomous rate but that the implementation of these new techniques by entrepreneurs can be explained by economic phenomena. Obviously, if the implementation of a new technique requires new capital equipment, as opposed to mere organizational change, increased productivity can be

[1] Preparation of an earlier draft of this essay was supported in part by the National Science Foundation under Grant GS-420 to the University of Chicago. K. J. Arrow, D. Cass, L. Hurwicz, and H. Uzawa contributed valuable comments and criticism.

transmitted only through new gross investment. In addition, Kaldor argues that for a capitalist economy the higher the relative rate of gross investment the higher is the degree of "technical dynamism." Technical dynamism is a mass measure of entrepreneurial psychology including the readiness to adopt new methods of production.

In the second model with endogenous technical change [3], Arrow concentrates upon the relation between learning and experience. Economic learning results in higher productivity; and cumulative gross investment is the measure of such economic experience. Therefore, in refining the technical progress function, Arrow explicitly postulates that per capita productivity is determined by accumulated gross investment. In this model, then, the production of new technical knowledge (invention) and the transmission and application of that knowledge (innovation) are treated as by-products in the production and adoption of new capital goods.

While it is doubtlessly true that technical change is related to gross investment both as a by-product of capital-goods production and as a vehicle for embodying new techniques in new capital equipment, it is also true that the rate of production of technical knowledge can be increased by increasing the allocation of economic resources explicitly devoted to inventive activity. In fact, much attention has been focused recently upon the economic aspects of invention or the process of creation of new technical knowledge.[2]

At least two peculiar properties of technical knowledge require special study. First, technical knowledge can be used by many economic units without altering its character. Thus, for the economy in which technical knowledge is a commodity, the basic premises of classical welfare economics are violated, and the optimality of the competitive mechanism is not assured. Typically, technical knowledge is very durable and the cost of transmission is small in comparison to the cost of production. Second, at least on the microeconomic level, the inventive process is characterized by extreme riskiness.

2. The Model

It has been argued that for an organized economy increases in technical knowledge are fundamentally related to the amount of resources devoted to inventive activity. In order to study the role of invention in economic growth, the model economy is divided into two sectors: a productive sector and an inventive sector.[3]

The homogeneous output of the productive sector $Y_2(t)$ is dependent upon

---

[2] Cf. [12], especially Arrow's contribution on pp. 609–625.

[3] This model is a two-sector extension of the one-sector model treated in [16]. In [16] I restricted myself to the special case where the production possibility frontier is a plane surface in the nonnegative orthant of the consumption-investment-invention space.

the amount of capital $K_2(t)$ and labor $L_2(t)$ currently devoted to that sector and upon the current level of technical knowledge $A(t)$.[4] Thus

$$Y_2(t) = \Phi_2[K_2(t), L_2(t), A(t)]. \tag{1}$$

The output of the productive sector either can be consumed or can be added to the existing capital stock. If capital is subject to evaporative decay at the given technical rate $\mu > 0$, then

$$\dot{K}(t) = s(t)Y_2(t) - \mu K(t), \tag{2}$$

where $K(t)$ is the current level of the capital stock, and $0 \leq s(t) \leq 1$ is the fraction of the productive output saved (and invested) at time t.

Abstracting from problems posed by uncertainty in aggregative invention, a deterministic relationship between the output of the inventive (or research) sector $Y_1(t)$ and the resources currently devoted to that sector is posited:

$$Y_1(t) = \Phi_1[K_1(t), L_1(t), A(t)]. \tag{3}$$

Of course, if $A(t)$ is interpreted as the current level of the stock of "social capital," then $Y_1(t)$ is current output of social capital. The stock of technical knowledge is considered to be subject to some rate $\rho \geq 0$ of instantaneous decay,

$$A(t) = Y_1(t) - \rho A(t). \tag{4}$$

For the case of positive ρ, Equation 4 should be understood as a long-run approximation to processes not explicitly treated in this model. For example, decay in technical knowledge is observed because of imperfect transmission of technical information from one generation of the labor force to the next.

It is assumed that the production functions defined in Equations 1 and 3 exhibit neoclassical constant returns to scale in capital and labor. That is, given A,

$$\lambda Y_j = \Phi_j(\lambda K_j, \lambda L_j, A) \qquad \text{for } K_j, L_j \geq 0,\ \lambda > 0; \qquad j = 1, 2.$$

In particular, assume that the production relations are multiplicative and of the homogeneous form

$$Y_j = AF_j(K_j, L_j) = \Phi_j(K_j, L_j, A) \qquad \text{for } j = 1, 2, \tag{5}$$

so that $F_j(\cdot)$ is positively linear-homogeneous in K_j and L_j.[5] Define $L(t)$ to

[4] Increases in efficiency are shared by all vintages of capital and labor; the embodiment problem is ignored.

[5] Thus, if $F_j(\cdot)$ is concave and increasing, $\Phi_j(\cdot)$ is an increasing semistrictly, quasi-concave function that is positively homogeneous of second degree. The specification of the production function given in Equation 5 is not crucial for the treatment of long-run behavior developed in section 5. However, specification of Equation 5 does

be the labor force inelastically offered for employment at time t. For an allocation of resources to be feasible at time t, it is required that

$$\begin{aligned} K_1(t) + K_2(t) &\leq K(t) \\ L_1(t) + L_2(t) &\leq L(t) \end{aligned} \qquad \text{where } K_1(t), K_2(t), L_1(t), L_2(t) \geq 0. \tag{6}$$

3. A Decentralized Economy

It is assumed that the productive sector is composed of many individual firms. The level of technical knowledge enters the firms' production functions as a pure public good of production. Hence, the competitive price of the output of the research sector is zero. This suggests the desirability of intervention in the market process.

Historically, intervention in behalf of inventive activity has taken two basic forms: first, the establishment of a legal device, the patent, designed to bestow property rights on certain of the outputs of the inventive process. The second form of intervention is that of direct nonmarket support of research and development. Universities have long played such a role in Western economies. In the United States, the Department of Agriculture has undertaken research activities since its inception. The Department of Commerce has initiated industrial research programs modeled after the agricultural research stations. The Department of Defense often uses the device of contracting research to private enterprises on a cost-plus-fixed-fee basis.

In the model decentralized economy, the rewards to capital and labor are paid in units of the output of the productive sector. The only form of intervention in the market process is an excise tax rate, $0 \leq \alpha < 1$, imposed upon the output of the productive sector. The revenue from the tax αY_2 is used for payment to the factors employed in the inventive sector. The research manager is assumed to maximize output of the inventive sector subject to this budget constraint.

Profits (after taxes) for the productive sector are

$$(1 - \alpha) Y_2 - wL_2 - rK_2,$$

where w is the wage rate of labor, and r is the rental rate of capital. If the individual production functions are identical and linear-homogeneous in capital and labor, then the result of the profit-maximization hypothesis is that

allow a simple aggregation in the productive sector that is congenial to the competitive hypothesis:

$$Y_2 \equiv \sum_i Y_2^i = A \sum_i F_2(K_2^i, L_2^i),$$

where, for example, K_2^i is the quantity of capital employed by the ith firm.

$$\left.\begin{aligned} \frac{\partial Y_2}{\partial K_2} &\leq \frac{r}{1-\alpha} \\ \frac{\partial Y_2}{\partial L_2} &\leq \frac{w}{1-\alpha} \end{aligned}\right\} \quad \text{with equality if } F_2(K_2, L_2) > 0. \tag{7}$$

To maximize output in the inventive sector, consider the Lagrangian form:

$$Y_1 + \theta(\alpha Y_2 - wL_1 - rK_1),$$

when $\theta \geq 0$ is a Lagrange multiplier. The conditions for maximization are

$$\left.\begin{aligned} \frac{\partial Y_1}{\partial K_1} &\leq \theta r \\ \frac{\partial Y_1}{\partial L_1} &\leq \theta w \end{aligned}\right\} \quad \text{with equality if } F_1(K_1, L_1) > 0. \tag{8}$$

Notice that if ω is defined to be the wage-rentals ratio (w/r), then if $F_j(K_j, L_j) > 0$,

$$\omega = \left(\frac{\partial F_j/\partial L_j}{\partial F_j/\partial K_j}\right) \qquad \text{for } j = 1, 2.$$

Define the usual per capita quantities:

$$k = \frac{K}{L}$$

and

$$k_j = \frac{K_j}{L_j}, \quad y_j = \frac{Y_j}{L}, \quad l_j = \frac{L_j}{L} \qquad \text{for } j = 1, 2.$$

The conditions for static equilibrium reduce to

$$wl_2 + rk_2l_2 = (1 - \alpha)y_2, \tag{9}$$

$$wl_1 + rk_1l_1 = \alpha y_2, \tag{10}$$

$$y_j = Af_j(k_j)l_j \qquad \text{for } j = 1, 2, \text{ where } f_j(k_j) = F_j(k_j, 1), \tag{11}$$

$$l_1 + l_2 = 1, \tag{12}$$

$$k_1l_1 + k_2l_2 = k, \tag{13}$$

$$\omega = \frac{f_j(k_j)}{f_j'(k_j)} - k_j \qquad \text{for } j = 1, 2. \tag{14}$$

Also assume that for each j, the function $f_j(k_j)$ is twice continuously differentiable for all k_j, and

$$\begin{gathered} f_j(k_j) > 0, \quad f_j'(k_j) > 0, \quad f_j''(k_j) < 0 \qquad \text{for } 0 < k_j < \infty; \\ f_j(0) = 0, \quad f_j(\infty) = \infty, \\ f_j'(0) = \infty, \quad f_j'(\infty) = 0. \end{gathered} \tag{15}$$

Then, the implicit relations $k_j(\omega)$ are well defined because

$$\frac{dk_j}{d\omega} = \frac{-[f_j'(k_j)]^2}{f_j(k_j)f_j''(k_j)} > 0, \qquad j = 1, 2. \tag{16}$$

Adding Equation 9 to Equation 10 and substituting in Equation 11,

$$Al_2 f_2(k_2) = w + rk.$$

But solving Equations 12 and 13 for l_2 yields

$$w + rk = \left(\frac{k - k_1}{k_2 - k_1}\right) Af_2(k_2).$$

However, from Equations 9 and 11,

$$f_2(k_2) = \frac{w + rk_2}{(1 - \alpha)A}.$$

Thus

$$w + rk = \left(\frac{w + rk_2}{1 - \alpha}\right)\left(\frac{k - k_1}{k_2 - k_1}\right).$$

Dividing by r yields

$$\omega + k = \left(\frac{\omega + k_2}{1 - \alpha}\right)\left(\frac{k - k_1}{k_2 - k_1}\right).$$

Let $Z = k + \omega$ and $Z_j = k_j + \omega$ for $j = 1, 2$.

$$Z(\omega) = \frac{Z_2 Z_1}{\alpha Z_2 + (1 - \alpha)Z_1} \qquad \text{where } 0 \leq \alpha < 1. \tag{17}$$

The right-hand side of Equation 17 takes all positive values and has a derivative everywhere greater than unity. Here $Z'(\omega)$ is identically unity, and hence, for given positive k, Equation 17 is uniquely solvable for ω, and *the greater the value of k, the greater the equilibrium value of ω.*[6]

Manipulation of Equation 17 yields theorems in comparative statics. For example, if the productive sector is always more (less) capital intensive than the inventive sector, (1) the higher the wage-rentals ratio, the higher (lower) is the equilibrium level of output of the inventive sector; (2) the higher the excise tax rate α, the higher (lower) is the wage-rentals ratio ω.

4. Static Efficiency and Optimal Taxation

Suppose that at a given moment in time the central planning board desires to maximize the expression

$$Y_2 + \lambda Y_1 \tag{18}$$

[6] My Equation 17 is formally equivalent to Equation 23 in [19]. Thus determination of static equilibrium in the model just outlined is equivalent to determination of static equilibrium in [19] with my excise tax rate playing the same role as Uzawa's average propensity to save.

subject to the resource constraint of Inequalities 6. Here λ is the social demand price of inventive output in terms of productive output.[7] By Conditions 15, maximization of Expression 18 requires full employment of resources; that is, Inequalities 6 must hold with equality. The maximum is achieved when $K_1 = K$, $L_1 = L$, *or* when $K_2 = K$, $L_2 = L$, *or* by solving the system

$$\begin{aligned} \frac{\partial Y_2}{\partial K_2} &= \lambda \frac{\partial Y_1}{\partial K_1}, \quad K_1 + K_2 = K, \\ \frac{\partial Y_2}{\partial L_2} &= \lambda \frac{\partial Y_1}{\partial L_1}, \quad L_1 + L_2 = L. \end{aligned} \tag{19}$$

Oniki and Uzawa [13] show that if the production functions satisfy Conditions 15, then there exist positive finite prices $\underline{\lambda}$ and $\bar{\lambda}$, with $0 < \underline{\lambda} \leq \bar{\lambda} < \infty$, such that for $\lambda \leq \underline{\lambda}$, maximization of Expression 18 requires specialization to production, and for $\lambda \geq \bar{\lambda}$, maximization of Expression 18 requires specialization to invention. For $\underline{\lambda} \leq \lambda \leq \bar{\lambda}$, first-order conditions of Equations 19 apply. By varying λ we obtain optimal outputs

$$Y_1 = Y_1(\lambda) \quad \text{and} \quad Y_2 = Y_2(\lambda),$$

[7] Suppose, for example, that the criterion of the planning board is to maximize the integral of discounted per capita consumption over some planning period $T > 0$,

$$\int_0^T (1 - s)AF_2(K_2, L_2)e^{-(n+\delta)t}\, dt,$$

where δ is the (constant) social discount rate, subject to initial conditions and terminal requirements. It is necessary for intertemporal optimality that the imputed value of gross national product

$$[(1 - s) + qs]Y_2 + vY_1$$

be maximized at every point in time. Here q is the social demand price of investment and v is the social demand price of invention. Thus for Expression 18,

$$\lambda = \frac{v}{(1 - s) + qs}.$$

Determination of the optimal trajectories for $q(t)$, $v(t)$, $s(t)$, and thus $\lambda(t)$, follows from the techniques of [14], and a treatment of the case where $k_1(\omega) = k_2(\omega)$ for all positive ω appears in [16]. In [16] the maximand was assumed to be a strictly concave function of per capita consumption and the production function was assumed to be strictly concave in its arguments. Under such conditions a long-run turnpike is found where A, k, q, and v are stationary and the net (of depreciation) socially valued marginal product of capital is equal to the net socially valued marginal product of technical knowledge. In the partially controlled economy, if long-run capital formation is lower (higher) than called for in the fully controlled economy, then long-run inventive activity should be higher (lower) in the partially controlled economy. It should be remarked that quasi-concavity of the $\Phi_j(\cdot)$ is not enough to ensure that Pontryagin's necessary conditions are sufficient. In fact, for $\Phi_j(\cdot)$ quasi-concave but not concave, I have found clearly nonoptimal programs satisfying the necessary conditions.

as nondecreasing and nonincreasing (respectively) upper-semicontinuous correspondences in λ.[8]

This is shown by simple construction. Choose social demand price λ^1 such that

$$Y_2{}^1 \in Y_2(\lambda^1) \quad \text{and} \quad Y_1{}^1 \in Y_1(\lambda^1).$$

Also choose social demand price λ^2 such that

$$Y_2{}^2 \in Y_2(\lambda^2) \quad \text{and} \quad Y_1{}^2 \in Y_1(\lambda^2).$$

Since Expression 18 is to be maximized,

$$Y_2{}^1 + \lambda^1 Y_1{}^1 \geq Y_2{}^2 + \lambda^1 Y_1{}^2, \tag{20}$$

and

$$Y_2{}^1 + \lambda^2 Y_1{}^1 \leq Y_2{}^2 + \lambda^2 Y_1{}^2. \tag{21}$$

Subtracting Inequality 21 from 20 yields

$$(\lambda^1 - \lambda^2)(Y_1{}^1 - Y_1{}^2) \geq 0,$$

which can be written as

$$\frac{\Delta Y_1}{\Delta \lambda} \geq 0, \quad \text{and similarly} \; \frac{\Delta Y_2}{\Delta \lambda} \leq 0, \tag{22}$$

where Δ is the finite difference operator.

Notice that for $0 < \alpha < 1$, the system of Equations 7 and 8 reduces to the system of Equations 19 when

$$\lambda = \frac{1}{\theta(1 - \alpha)}, \tag{23}$$

that is, if the social demand price of inventive output is equal to the implicit (supply) valuation of inventive output. If Equation 23 holds, factor payments in the inventive sector are

$$\lambda(1 - \alpha)L_1 \frac{\partial Y_1}{\partial L_1} + \lambda(1 - \alpha)K_1 \frac{\partial Y_1}{\partial K_1}.$$

When the inventive sector equates factor payments to revenues

$$\frac{\alpha}{1 - \alpha} = \frac{\lambda L_1(\partial Y_1/\partial L_1) + \lambda K_1(\partial Y_1/\partial K_1)}{Y_2}.$$

But by Euler's theorem

$$\frac{\alpha}{1 - \alpha} = \frac{\lambda Y_1(\lambda)}{Y_2(\lambda)}. \tag{24}$$

For $0 < \alpha < 1$, the left-hand side of Equation 24 is a strictly increasing function of α with range $(0, \infty)$. For $\underline{\lambda} < \lambda < \bar{\lambda}$, the right-hand side of

[8] Cf. p. 17 in [5].

Equation 24 is a nondecreasing correspondence of λ with range $(0, \infty)$. Given $\alpha \in (0, 1)$, Equation 24 is solvable for $\lambda \in (\underline{\lambda}, \bar{\lambda})$.

Thus *for any tax rate* $0 < \alpha < 1$, *the decentralized economy is efficient.* The trivial case $\alpha = 0$ corresponds to maximization of Expression 18 for the case $\lambda \leq \underline{\lambda}$. For $\alpha = 1$, allocation of resources between the two sectors is not uniquely determined.

Next define the implicit (supply) price of invention by

$$p(\alpha) = \frac{f_2'\{k_2[\omega(\alpha)]\}}{f_1'\{k_1[\omega(\alpha)]\}} = \frac{1}{\theta(1-\alpha)} \qquad \text{for } 0 < \alpha < 1. \tag{25}$$

But notice by Equation 17,

$$\lim_{\alpha \to 0} k_2[\omega(\alpha)] = k, \quad \lim_{\alpha \to 1} k_1[\omega(\alpha)] = k,$$

and thus

$$\lim_{\alpha \to 0} p(\alpha) = \bar{\lambda}, \quad \lim_{\alpha \to 1} p(\alpha) = \underline{\lambda}.$$

Logarithmic differentiation of Equation 25 yields

$$\frac{1}{p}\frac{dp}{d\omega} = \frac{1}{k_1 + \omega} - \frac{1}{k_2 + \omega} \gtrless 0 \qquad \text{as } k_2 \gtrless k_1,$$

and differentiation of Equation 17 yields

$$\frac{\partial \omega}{\partial \alpha} = \frac{(1/Z_1) - (1/Z_2)}{(\alpha Z_1'/Z_1^2) + [(1-\alpha)Z_2'/Z_2^2] - (1/Z^2)} \gtrless 0 \qquad \text{as } k_2 \gtrless k_1.$$

Thus

$$\frac{\partial p}{\partial \alpha} > 0 \qquad \text{for } 0 < \alpha < 1.$$

Hence, *given the social demand price for inventive output in terms of productive output,* $\underline{\lambda} < \lambda < \bar{\lambda}$, *the optimum excise tax rate* $0 < \alpha < 1$ *can be determined by solving Equation 24.* For $\lambda \leq \underline{\lambda}$, optimality requires that in the decentralized economy the excise tax rate be set equal to zero. For $\lambda \geq \bar{\lambda}$, optimality requires the economy to be centralized with all factors of production to be allocated to the research manager.

5. Long-Run Behavior of the Economy with Constant Rates of Savings and Taxation

Consider the stylized Western economy in which static equilibrium is determined by Equations 9 through 14. Assume that the excise tax rate α and the savings fraction s are institutionally given and fixed through time with $0 < \alpha < 1$ and $0 < s < 1$. Capital accumulation is determined by Equation 2 and the change over time of the stock of technical knowledge proceeds in accordance with Equation 4. In order to simplify the analysis, assume that the working population is stationary so that we can set $L = 1$ without loss in generality. Initial endowments of resources $A(0)$, $K(0)$, and $L(0) = 1$ are given.

Rewriting Equation 2 in per capita terms yields

$$\dot{k} = sAl_2 f_2(k_2) - \mu k. \tag{26}$$

The capital stock is stationary ($\dot{k} = 0$) if and only if

$$A = \frac{\mu k}{sl_2 f_2(k_2)}. \tag{27}$$

Equation 27 defines a curve in the positive quadrant of the $k - A$ plane.[9]

Rewriting Equation 4 yields

$$\dot{A} = A[l_1 f_1(k_1) - \rho]. \tag{28}$$

With $A > 0$, the stock of technical knowledge is stationary only if the stock of physical capital is such that

$$l_1 f_1(k_1) = \rho \tag{29}$$

is satisfied. By Equations 15 the left-hand side of Equation 29 tends to zero as k tends to zero. Assume, for example, that the left-hand side of Equation 29 is an analytic function of k and tends to infinity as k tends to infinity. Then there exists $0 < \bar{k} < \infty$ such that

$$l_1 f_1(k_1) > \rho \qquad \text{for } k > \bar{k}.$$

By a classic theorem of analytic function theory,[10] solutions to Equation 29 cannot be dense. Therefore by the Bolzano-Weierstrass theorem, the number of distinct solutions to Equation 29 is finite and odd.

The phase diagram (Figure 1) depicts the long-run behavior of the system of differential Equations 26 and 28 for the case when there are exactly three solutions k^*, k^{**}, k^{***} to Equation 29. Let

$$0 < k^* < k^{**} < k^{***} < \infty.$$

There are then exactly three equilibrium points. The points (k^*, A^*) and

[9] In fact, if the elasticity of substitution between capital and labor is never less than unity in the inventive sector *or* if invention is always more capital intensive than production, then

$$\left.\frac{dA}{dk}\right|_{\dot{k}=0} = \frac{A\mu}{s}\left[\frac{y_2 - k(dy_2/dk)}{y_2^2}\right] > 0.$$

The proof of this proposition is in the appendix to this essay.

[10] Here $l_1 f_1$ is assumed to be an analytic function of k and the constant ρ is a trivial analytic function of k. Hence, if $l_1 f_1$ equals ρ on a set that has an accumulation point

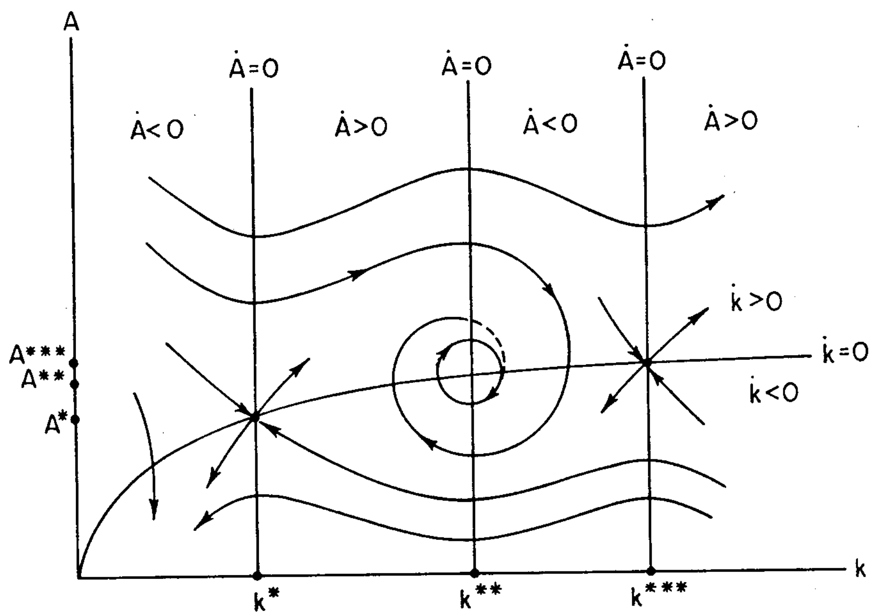

FIGURE 1. Multiple equilibriums.

(k^{***}, A^{***}) are saddle points.[11] Locally, the point (k^{**}, A^{**}) is either stable or (by the Poincaré-Bendixon theorem) there exists a limit cycle forming a periodic orbit about (k^{**}, A^{**}). The limit cycle case is illustrated in Figure 1.

The case where

$$\frac{dy_1}{dk} > 0 \qquad \text{for } k > 0 \tag{30}$$

is of special interest.[12] If Inequality 30 holds, the solution to Equation 29 is unique, and therefore there is a unique equilibrium for the system of differential Equations 26 and 28. This case is illustrated in the phase diagram of Figure 2. The unique equilibrium (k^*, A^*) is a saddle point. Thus the k–A plane is divided by a "razor's edge." For initial endowments of physical capital and technical knowledge below this line, the economy "decays." For initial endowments above this line, the economy "explodes." In the general case, there exists the possibility that the economy tends to a technological trap or periodic orbit,[13] for example, the point (k^{**}, A^{**}) in Figure 1.

[11] Remembering that $f_j(\cdot)$ is twice continuously differentiable.

[12] A sufficient condition for Inequality 30 to hold is (1) invention is always more capital intensive than production, or (2) the elasticity of substitution in production is never less than unity. Again, the proof of this proposition appears in the appendix to this essay.

[13] In the recent literature of economic growth theory it is usual to examine a model for global stability. It has been found for certain models that their ultimate long-run behavior

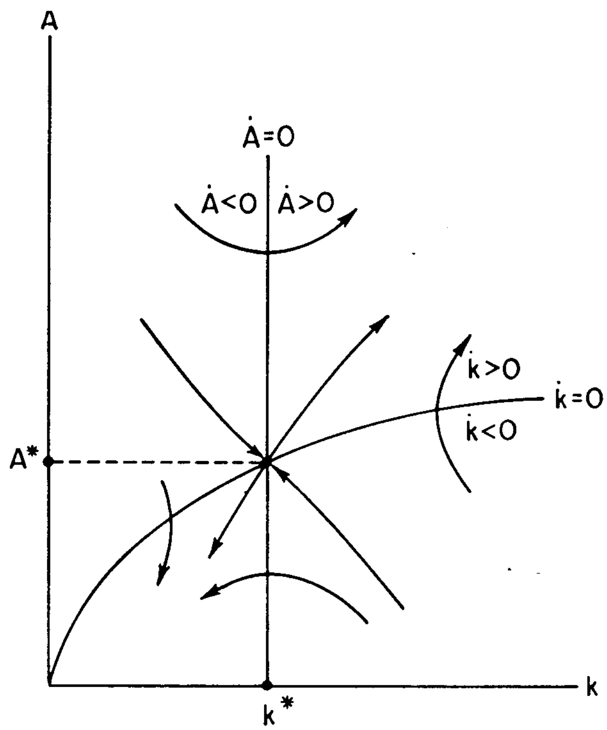

FIGURE 2. Unique equilibriums.

6. Concluding Comments

The model just discussed focuses upon the role of invention in economic growth. In order to simplify the analysis, certain important phenomena are ignored. Among other things, the effects of a growing labor force and the process of transmission of knowledge within the economy are ignored. Nonetheless, the model is sufficiently rich at least to suggest explanations for some economic problems.

The post-World War II experience of Germany and Japan provides an instructive example from recent economic history. It could be argued that although large amounts of their physical capital were destroyed during the

is independent of initial endowments. See, e.g., [17] and [19]. While such stability (if it exists) is certainly an interesting property of any model, it should not be considered an essential property for a growth model. In fact, Maruyama [11] argues just the opposite: that social systems are basically morphogenetic rather than morphostatic. The notion and the usefulness of the concept of stability in the engineering sciences is treated in [4].

war, their stocks of technical knowledge remained large enough with respect to the remaining stocks of physical capital to ensure the sustenance of explosive growth.[14] Thus the war-torn economies staged "miraculous" recoveries while certain nonbelligerent but impoverished economies remained impoverished.

This example raises an important question: Why do not technically backward economies freely adopt the techniques developed in the advanced economies? A certain amount of such "copying" does occur, but, contrary to the model presented here, in real life endowments of productive factors are not homogeneous, and knowledge that is useful to one country may not be useful for production in another country. Even then, transmission of technical information (education, innovation, and so on) is certainly not a costless activity. The role of transmission of technical information in the process of economic development is a topic that is worthy of further investigation.

APPENDIX: COMPARATIVE STATICS FOR THE TWO-SECTOR MODEL

In this appendix, certain simple propositions in comparative statics are developed for the two-sector model. I was led to the study of these propositions because of their relevance to the direction and stability of long-run growth in the model of inventive activity and capital accumulation. These propositions are of some independent interest and therefore a separate treatment is warranted.[15]

Consider the miniature two-sector Walrasian equilibrium system given by Equations 9 through 14. Using the supply price p defined in Equation 25, gross national income per capita y is given by

$$y = y_2 + py_1. \tag{A.1}$$

Demand for output can be rewritten as

$$py_1 = \alpha y. \tag{A.2}$$

[14] My colleague P. N. Rosenstein-Rodan stresses also the importance in the recovery process of the remaining stocks of *physical* social capital. He has told me that because of exceptional circumstances the Neapolitan sewers were devastated by Allied bombing. This, he argues, was sufficient to cause Naples to require enormous outside aid in order to "get back on its feet."

[15] The pioneer work in two-sector comparative statics seems to be that of Rybczynski [15] whose analysis is in terms of the Samuelson-Stolper box diagram. I am indebted to J. Wise for the reference to Rybczynski's note.

Setting $A = 1$ for convenience, and combining Equations A.1 and A.2 with Equation 11 yields the basic equation

$$\left(\frac{1-\alpha}{\alpha}\right)\frac{f_2'(k_2)}{f_1'(k_1)} = \left(\frac{k-k_1}{k_2-k}\right)\frac{f_2(k_2)}{f_1(k_1)}. \tag{A.3}$$

Logarithmically differentiating both sides of Equation A.3 with respect to k, yields the total derivative

$$\frac{d\omega}{dk} = \frac{\dfrac{k_2-k_1}{(k-k_1)(k_2-k)}}{\dfrac{dk_2}{d\omega}\left(\dfrac{f_2''}{f_2'}-\dfrac{f_2'}{f_2}+\dfrac{1}{k_2-k}\right)+\dfrac{dk_1}{d\omega}\left(\dfrac{f_1'}{f_1}-\dfrac{f_1''}{f_1'}+\dfrac{1}{k-k_1}\right)}. \tag{A.4}$$

Because of Equations 14 and 16, Equation A.4 can be rewritten as

$$\frac{d\omega}{dk} = \frac{\dfrac{k_2-k_1}{(k-k_1)(k_2-k)}}{\dfrac{dk_2}{d\omega}\left[\dfrac{k+\omega}{(k_2-k)(k_2+\omega)}\right]+\dfrac{dk_1}{d\omega}\left[\dfrac{k+\omega}{(k-k_1)(k_1+\omega)}\right]+\left[\dfrac{k_2-k_1}{(k_2+\omega)(k_1+\omega)}\right]}. \tag{A.5}$$

For $k_2 \neq k \neq k_1$, the numerator and the denominator of the right-hand side of Equation A.5 are seen to agree in sign, and therefore $(d\omega/dk) > 0$ for $0 < k < \infty$. This is the proposition (Uzawa) that if the demand for output is such that the marginal propensity to consume equals the average propensity to consume, then *the higher the endowment of a factor of production, the lower is the equilibrium level of the relative reward to that factor.* Also given $0 < k < \infty$, the equilibrium value of ω is uniquely determined.

Next, observe the *direct* effect of differing factor endowments upon the equilibrium composition of output. From Equation 11,

$$\frac{\partial y_1}{\partial k} = \frac{-f_1(k_1)}{k_2-k_1}, \qquad \frac{\partial y_2}{\partial k} = \frac{f_2(k_2)}{k_2-k_1}. \tag{A.6}$$

The partial derivatives in Equations A.6 are independent of demand, and thus we have the proposition (Rybczynski): *If the rates of substitution in production are fixed, that is,* $(d\omega/dk) = 0$, *then the higher the endowment of a factor of production, the higher (lower) is the equilibrium level of production of*

Logarithmic differentiation of Equation 11 yields

$$\frac{1}{y_1}\frac{dy_1}{d\omega} = \frac{dk_2}{d\omega}\left[\frac{k - k_1}{(k_2 - k)(k_2 - k_1)}\right] + \frac{dk_1}{d\omega}\left[\frac{k_2 + \omega}{(k_1 + \omega)(k_2 - k_1)}\right] \gtrless 0 \quad \text{(A.7)}$$

as $k_2 \gtrless k \gtrless k_1$. Similarly $(1/y_2)(dy_2/d\omega) \lesseqgtr 0$ as $k_2 \gtrless k \gtrless k_1$.

Since $(d\omega/dk) > 0$, we have for the system of Equations 9 through 14 that *the direct effect (Equation A.6) of varying factor endowment upon equilibrium levels of output is opposite in sign to the indirect effect (Equation A.7).*

The *total effect* is the sum of the direct effect and the indirect effect:

$$\frac{dy_1}{dk} = \frac{\partial y_1}{\partial k} + \frac{dy_1}{d\omega}\frac{d\omega}{dk}, \qquad \frac{dy_2}{dk} = \frac{\partial y_2}{\partial k} + \frac{dy_2}{d\omega}\frac{d\omega}{dk}. \quad \text{(A.8)}$$

Because of Equations A.6 and A.7, the first of Equations A.8 can be rewritten as

$$\frac{dy_1}{dk} = \frac{\partial y_1}{\partial k}\left(1 - \frac{N}{D}\right), \quad \text{(A.9)}$$

where N is defined by

$$N = \frac{dk_2}{d\omega}\left(\frac{k - k_1}{k_2 - k_1}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2 - k}{k_2 - k_1}\right)\left(\frac{k_2 + \omega}{k_1 + \omega}\right) > 0, \quad \text{(A.10)}$$

and D is defined by

$$D = \frac{dk_2}{d\omega}\left(\frac{k - k_1}{k_2 - k_1}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2 - k}{k_2 - k_1}\right)\left(\frac{k + \omega}{k_1 + \omega}\right) + \frac{(k - k_1)(k_2 - k)}{k_2 - k_1}\left[\left(\frac{f_2''}{f_2'} - \frac{f_2'}{f_2}\right)\frac{dk_2}{d\omega} - \frac{f_1''}{f_1'}\frac{dk_1}{d\omega}\right]. \quad \text{(A.11)}$$

Applying Equations 14 and 16 to Equation A.11 gives

$$D = \frac{dk_2}{d\omega}\left(\frac{k - k_1}{k_2 - k_1}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2 - k}{k_2 - k_1}\right)\left(\frac{k + \omega}{k_1 + \omega}\right) + \frac{(k - k_1)(k_2 - k)}{k_2 - k_1}\left[\frac{1}{k_1 + \omega} - \frac{1 + (dk_2/d\omega)}{k_2 + \omega}\right] > 0. \quad \text{(A.12)}$$

Consider the case where $k_1 > k > k_2$. From Equations A.6 we have $(\partial y_1/\partial k) > 0$ and therefore $(dy_1/dk) > 0$ if and only if $D > N$. Examine the right-hand sides of Equations A.10 and A.12. The first terms are identical; for $k > k_2$ the second term in Equation A.12 is greater than the second term in Equation A.10. For $k_1 > k_2$, the third term in Equation A.12 is positive. Hence, when $k_1 > k > k_2$, then $D > N > 0$ or $0 < (N/D) < 1$.

For the two-sector economy (Equations 9 through 14), *the higher the endowment of a factor of production, the higher is the equilibrium level of output of the commodity using relatively much of that factor.*

Consider the reverse factor-intensity case, $k_2 > k > k_1$. Here (dy_1/dk) is positive if and only if $(D - N) < 0$, or subtracting Equation A.10 from Equation A.11,

$$(k - k_1)\frac{dk_2}{d\omega}\left(\frac{f_2''}{f_2'} - \frac{f_2'}{f_2}\right) - \frac{dk_1}{d\omega}\left[\frac{(k_2 - k)f_1'}{f_1} + \frac{(k - k_1)f_1''}{f_1'}\right] < 0. \quad \text{(A.13)}$$

Multiplying both sides of Inequality A.13 by (ω/k_1k_2) and substituting from Equations 14 and 16 yields

$$\frac{(k - k_1)\sigma_2}{k_1(k_2 + \omega)} + \left(\frac{\omega}{k_1k_2}\right)\left(\frac{k - k_1}{k_2 + \omega}\right) + \frac{(k_2 - k)\sigma_1}{k_2(k_1 + \omega)} > \left(\frac{\omega}{k_1k_2}\right)\left(\frac{k - k_1}{k_1 + \omega}\right), \quad \text{(A.14)}$$

where σ_j $(j = 1, 2)$ is the elasticity of substitution between factors in the jth sector. This basic property of production functions was introduced by Hicks and refined by Allen.[16] The elasticity of substitution can be written as

$$\sigma_j(\omega) = \frac{\omega}{k_j}\frac{dk_j}{d\omega} \qquad \text{for } j = 1, 2.$$

Rearranging Inequality A.14 gives

$$\sigma_2 > \frac{\omega}{k_1 + \omega} - \frac{\omega k_1}{\omega k_2 + k_1k_2} - \frac{k_1(k_2 + \omega)(k_2 - k)\sigma_1}{k_2(k_1 + \omega)(k - k_1)}. \quad \text{(A.15)}$$

From Inequality A.15 a simple *sufficient* condition for (dy_1/dk) to be positive is that $\sigma_2 \geq 1$. Thus, *if the elasticity of substitution in the production of commodity two (one) is greater than or equal to unity, then the higher the endowment of either factor of production, the higher is the equilibrium level of output of commodity one (two).*

It is instructive to study the special case where the production functions (Equations 1 and 3) are linear in logarithms. For this case, we can write

$$\begin{aligned} f_1(k_1) &= k_1{}^a \qquad \text{where } 0 < a < 1, \\ f_2(k_2) &= k_2{}^b \qquad \text{where } 0 < b < 1. \end{aligned} \quad \text{(A.16)}$$

Applying Equation 14 to Equations A.16 yields

$$k_1 = \frac{a\omega}{1 - a} \quad \text{and} \quad k_2 = \frac{b\omega}{1 - b}. \quad \text{(A.17)}$$

[16] Cf. pp. 117, 245 in [6], pp. 341–343 in [2], and [7].

Substituting Equations A.16 and A.17 in Equation A.3 yields

$$\omega = \left[\frac{\alpha(1 - a) + (1 - \alpha)(1 - b)}{\alpha a + (1 - \alpha)b}\right]k. \tag{A.18}$$

From Equations A.17 and A.18,

$$k_1 = \gamma_1 k \quad \text{and} \quad k_2 = \gamma_2 k,$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are constants fixed upon specification of the parameters α, a, b. Notice also that

$$l_1 = \frac{k_2 - k}{k_2 - k_1} = \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}$$

and

$$l_2 = \frac{k - k_1}{k_2 - k_1} = \frac{1 - \gamma_1}{\gamma_2 - \gamma_1}.$$

Therefore $0 < l_1 < 1$ and $0 < l_2 < 1$ are, in the Cobb-Douglas case, fixed constants. Hence if production satisfies Equations 11 through 14 and Equations A.16, and if demand satisfies Equations A.1 and A.2, then (dy_1/dk) and (dy_2/dk) are positive for all $k > 0$.

In the study of the model of inventive activity and capital accumulation,[17] one is interested in the sign of an expression that is equivalent to

$$y_2 - k\frac{dy_2}{dk}, \tag{A.19}$$

which can be rewritten as

$$\frac{f_2(k_2)}{D(k_2 - k_1)}(k\tilde{N} - k_1 D), \tag{A.20}$$

where by Equation 14, $\tilde{N}$ is given by

$$\tilde{N} = \frac{dk_2}{d\omega}\left(\frac{k_1 + \omega}{k_2 + \omega}\right)\left(\frac{k - k_1}{k_2 - k_1}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2 - k}{k_2 - k_1}\right) > 0 \qquad \text{for } k_2 \neq k \neq k_1,$$

and where $D > 0$ is defined by Equation A.11.

Form the expression

$$\begin{aligned} k\tilde{N} - k_1 D = \frac{k - k_1}{k_2 - k_1}\Bigg\{&\frac{dk_2}{d\omega}\left[k - k_1 - \frac{k(k_2 - k_1)}{k_2 + \omega} + \frac{k_1(k_2 - k)}{k_2 + \omega}\right] \\ &+ \frac{dk_1}{d\omega}\left(\frac{k_2 - k}{k - k_1}\right)\left[k - k_1 - \frac{k_1(k - k_1)}{k_1 + \omega}\right] \\ &+ \frac{k_1(k_2 - k)}{k_2 + \omega} - \frac{k_1(k_2 - k)}{k_1 + \omega}\Bigg\}. \end{aligned} \tag{A.21}$$

[17] Cf., e.g., p. 76 of the text.

The right-hand side of Equation A.21 can be rewritten as

$$\frac{(k - k_1)(k_2 - k)}{k_2 - k_1}\left[\frac{dk_2}{d\omega}\left(\frac{k - k_1}{k_2 - k}\right)\left(1 - \frac{k_2}{k_2 + \omega}\right) + \frac{dk_1}{d\omega}\left(1 - \frac{k_1}{k_1 + \omega}\right) + k_1\left(\frac{1}{k_2 + \omega} - \frac{1}{k_1 + \omega}\right)\right]. \tag{A.22}$$

Notice that for the case $k_1 > k > k_2$, Expression A.22 is negative. Therefore, if $k_1 > k > k_2$, then Expressions A.19 and A.20 are positive.

For the reverse case $k_2 > k > k_1$, Expression A.22 tells us that $k\tilde{N} > k_1 D$ if and only if

$$\frac{dk_2}{d\omega}\left(\frac{k - k_1}{k_2 - k}\right)\left(\frac{\omega}{k_2 + \omega}\right) + \frac{dk_1}{d\omega}\left(\frac{\omega}{k_1 + \omega}\right) > \frac{k_1(k_2 - k_1)}{(k_1 + \omega)(k_2 + \omega)}. \tag{A.23}$$

Dividing Inequality A.23 by k_1k_2 yields

$$\sigma_1 > \frac{k_2}{k_2 + \omega} - \frac{k_1}{k_2 + \omega} - \left(\frac{k - k_1}{k_2 - k}\right)\left(\frac{k_1 + \omega}{k_2 + \omega}\right)\frac{k_2}{k_1}\sigma_2. \tag{A.24}$$

Therefore, a simple *sufficient* condition for Expression A.19 to be positive is for $k_1 > k_2$ or $\sigma_1 \geq 1$.

References

1. Ahlfors, L. V., *Complex Analysis*, New York: McGraw-Hill Book Company, 1953.
2. Allen, R. G. D., *Mathematical Analysis for Economists*, New York: St. Martin's Press, Inc., 1962.
3. Arrow, K. J., "The Economic Implications of Learning by Doing," *Review of Economic Studies*, Vol. 29 (June 1962), pp. 155–173.
4. Cunningham, W. J., "The Concept of Stability," *The American Scientist*, Vol. 51 (December 1963), pp. 425–436.
5. Debreu, G., *Theory of Value*, New York: John Wiley & Sons, Inc., 1959.
6. Hicks, J. R., *The Theory of Wages*, London: Macmillan and Company, 1932.
7. Hicks, J. R., and R. G. D. Allen, "A Reconsideration of the Theory of Value," *Economica* (N.S.), Vol. 1 (May 1934), pp. 196–217.
8. Kaldor, N., "A Model of Economic Growth," *Economic Journal*, Vol. 67 (December 1957). Reprinted in *Essays on Economic Stability and Growth*, New York: Free Press of Glencoe, Inc., 1960, pp. 256–300.
9. Kaldor, N., "Capital Accumulation and Economic Growth," in F. A. Lutz and D. C. Hague (eds.), *The Theory of Capital*, New York: St. Martin's Press, Inc., 1961, pp. 177–222.
10. Kaldor, N., and J. A. Mirrlees, "A New Model of Economic Growth," *Review of Economic Studies*, Vol. 29 (June 1962), pp. 174–192.
11. Maruyama, M., "Morphogenesis and Morphostasis," *Methodos*, Vol. 12, No. 48 (1960).

12. National Bureau of Economic Research, *The Rate and Direction of Inventive Activity*, Princeton: Princeton University Press, 1962.
13. Oniki, H., and H. Uzawa, "Patterns of Trade and Investment in a Dynamic Model of International Trade," *Review of Economic Studies*, Vol. 32, No. 1 (January 1965), pp. 15–38.
14. Pontryagin, L. S., *et al.*, *The Mathematical Theory of Optimal Processes*, New York and London: Interscience Publishers, Inc., 1962.
15. Rybczynski, T. N., "Factor Endowment and Relative Commodity Prices," *Economica* (N.S.), Vol. 22 (November 1955), pp. 336–341.
16. Shell, K., "Toward a Theory of Inventive Activity and Capital Accumulation," *American Economic Review*, Vol. 56, No. 2 (May 1966), pp. 62–68.
17. Solow, R. M., "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, Vol. 32 (1956), pp. 65–94.
18. Swan, T., "Economic Growth and Capital Accumulation," *Economic Record*, Vol. 66 (1956), pp. 334–361.
19. Uzawa, H., "On a Two-Sector Model of Economic Growth II," *Review of Economic Studies*, Vol. 30 (June 1963), pp. 105–118.