I

Optimal Programs of Capital Accumulation for an Economy in which there is Exogenous Technical Change[1]

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1. The One-Sector Model

Recall the aggregative economic growth model of Solow [13] and Swan [14]. In the model economy, there are two factors of production, capital and labor, that are combined to produce a single homogeneous output. At any instant in time a fraction of this homogeneous output may be allocated to consumption and the remaining fraction allocated to investment in capital accumulation. Once invested the capital stock is bolted down in the sense that in itself it is not a good that is fit for consumption.

If $K(t)$ and $L(t)$ denote the currently existing stocks of capital and labor, then the current rate of output $Y(t)$ can be expressed by

$$Y(t) = A(t)F[K(t), L(t)], \tag{1}$$

where $A(t)$ is a measure of the current level of technical knowledge. Here $F[\cdot]$ is the neoclassical production function exhibiting constant returns to scale in capital and labor; that is,

$$\lambda Y = F[\lambda K, \lambda L] \qquad \text{for } K, L \geq 0 \text{ and } \lambda > 0. \tag{2}$$

[1] Research for this paper was undertaken in 1963–1964 when I was Woodrow Wilson Dissertation Fellow at Stanford University. Preparation of the manuscript was supported in part by a Ford Foundation faculty research grant to the Department of Economics at Massachusetts Institute of Technology. I am indebted to K. J. Arrow, D. Cass, F. M. Fisher, P. A. Samuelson, R. M. Solow, and H. Uzawa for helpful suggestions.

Let $C(t)$ and $Z(t)$ denote the current rates of consumption and investment; let $0 \leq s(t) \leq 1$ denote the fraction of current output that is being saved (and invested). This yields the national income identities

$$Y \equiv C + Z \equiv (1 - s)Y + sY.$$

If capital is subject to evaporative decay at the constant rate $\mu > 0$, then growth of the capital stock is specified by the differential equation

$$\dot{K}(t) = s(t)A(t)F[K(t), L(t)] - \mu K(t). \tag{3}$$

Assume that $N(t)$ is the current size of the population for the entire society. Assume that population growth is independent of the economic variables, in particular that

$$\dot{N}(t) = nN(t),$$

where n is a constant. Assume further that the number of able-bodied workers is a stationary fraction $0 < \alpha < 1$ of the total population. If the central planning board requires all able-bodied citizens to be workers, $L(t) = \alpha N(t)$, and thus

$$\dot{L}(t) = nL(t). \tag{4}$$

Assume that technical change proceeds at an autonomous fixed relative rate ρ,[2]

$$\dot{A}(t) = \rho A(t). \tag{5}$$

The problem is to characterize the program of capital accumulation that is consistent with the system in Equations 1 through 5 and maximizes some suitable criterion (or welfare) functional while satisfying appropriate initial conditions and terminal requirements, if any. If $\delta > 0$ is the planning board's (constant) rate of time discount for per capita consumption, then the problem is equivalent to that of maximizing the expression

$$\int_0^T \frac{C(t)}{L(t)} e^{-\delta t}\, dt.$$

The maximand is constrained by the system of Equations 1 through 5 and by the given initial conditions $K(0) = K_0$, $L(0) = L_0$, and $A(0) = A_0$. It may also be required (for instance, for reasons of national prestige) that at the terminal planning date $0 < T \leq \infty$, the capital-labor ratio is not less than some prescribed target, or that $K(T)/L(T) \geq k_T$.

[2] The model has three interpretations: (1) For $\rho = 0$, it is the one-sector model with an unchanging menu of techniques. (2) For $\rho > 0$, it is the model with positive technical progress. (3) For $\rho < 0$, it can be interpreted as a special case of my model [12]. Then A is interpreted as the stock of social capital. Under a libertarian administration (no support of social goods production), A declines following the equation $\dot{A} = \rho A$ where $(-\rho)$ is the instantaneous rate of depreciation of social capital.

First, define the usual per capita quantities:

Output per worker: $y(t) = Y(t)/L(t)$.
Aggregate capital-labor ratio: $k(t) = K(t)/L(t)$.
Consumption per worker: $c(t) = C(t)/L(t)$.
Investment per worker: $z(t) = Z(t)/L(t)$.

The problem reduces to the following problem in miniature form:

To maximize:

$$\int_0^T c(t)e^{-\delta t}\,dt \tag{6}$$

subject to the constraints:

$$\dot{k}(t) = s(t)y(t) - \lambda k(t), \tag{7}$$

$$y(t) = e^{\rho t}f[k(t)], \tag{8}$$

$$0 \le s(t) \le 1, \tag{9}$$

$$k(0) = k_0, \quad \text{and} \quad k(T) \ge k_T, \tag{10}$$

where δ, $\lambda \equiv n + \mu$, k_0, k_T are given constants, and $s(t)$ is some measurable control (or policy) variable to be chosen. Units of measurement have been chosen such that $A(0) = 1$ and therefore $A(t) = e^{\rho t}$.

The expression $F(k, 1)$ is replaced by the usual shorthand expression $f(k)$, which is assumed to be thrice continuously differentiable. Production satisfies the following neoclassical conditions:

$$\begin{aligned} &f(k) > 0, \quad f'(k) > 0, \quad f''(k) < 0, \qquad \text{for } 0 < k < \infty, \\ &f(0) = 0, \quad f'(0) = \infty, \quad f(\infty) = \infty, \qquad f'(\infty) = 0. \end{aligned} \tag{11}$$

The foregoing problem is solved by employing the "maximum principle" of Pontryagin *et al.* [8]. Introduce the Hamiltonian form

$$(1 - s)e^{(\rho - \delta)t}f(k) + qe^{-\delta t}[se^{\rho t}f(k) - \lambda k].$$

If a program $[k(t), s(t);\ 0 \le t \le T]$ is optimal,[3] then there exists a continuous function $q(t)$ such that

$$\dot{k}(t) = s(t)e^{\rho t}f[k(t)] - \lambda k(t), \tag{12}$$

with initial condition $k(0) = k_0$,

$$\dot{q}(t) = (\delta + \lambda)q(t) - \{[1 - s(t)] + q(t)s(t)\}e^{\rho t}f'[k(t)], \tag{13}$$

$$s(t) \text{ maximizes } [1 - s(t) + q(t)s(t)] \text{ subject to } 0 \le s(t) \le 1, \tag{14}$$

and s is a piecewise continuous function of t,

$$e^{-\delta T}q(T)[k(T) - k_T] = 0. \tag{15}$$

[3] Cf. [8], especially Theorem 3 (p. 50) and also pp. 108–114, 189–191.

For convenience set

$$\gamma = \max_{0 \le s \le 1} [(1 - s) + qs] = \max(1, q).$$

Notice that $q(t)$ has the interpretation of the social demand price of a unit of investment in terms of a currently forgone unit of consumption. Therefore, differential Equation 13 may be interpreted as the requirement of perfect foresight. In a competitive economy, for example, the change in the price of a unit of capital should compensate a *rentier* for loss due to depreciation and for "abstinence" net of any rewards from the employment of that unit of capital. Transversality Condition 15 states that at the target date the target requirement (Equation 10) must hold with equality or the present value of the target demand price of investment must be zero.

Next, it is required to study the singular solutions of differential Equation 13. Notice that $\dot{q} = 0$ if and only if

$$q = \frac{\gamma e^{\rho t} f'(k)}{\delta + \lambda}. \tag{16}$$

Equation 16 reduces to

$$e^{\rho t} f'(k_t) = \delta + \lambda \qquad \text{for case } q \geq 1, \tag{17}$$

and

$$q_t = \frac{e^{\rho t} f'(k_t)}{\delta + \lambda} \qquad \text{for case } q \leq 1. \tag{18}$$

If the production function satisfies Conditions 11, it is well known that for any instant of time Equation 17 is uniquely solvable in k_t. Call the solution to Equation 17, k_t^*. Determination of k_t^* is shown in Figure 1. Here $\tilde{k}_t$ is the maximum sustainable capital-labor ratio when technology is held fixed.

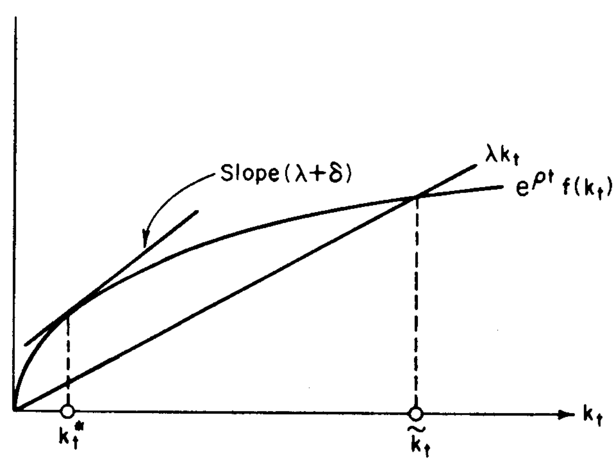

FIGURE 1. Determination of k_t^* and $\tilde{k}_t$.

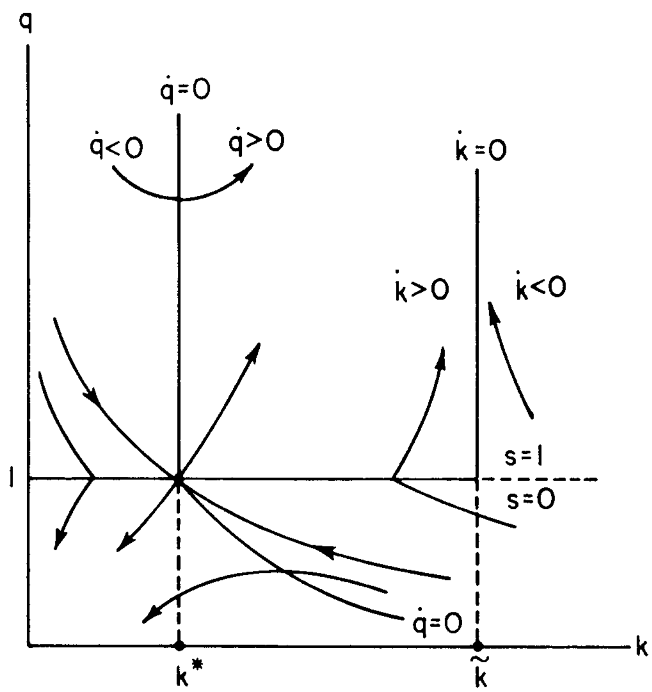

FIGURE 2. Phase diagram for the case $\rho = 0$.

It is easily shown that for fixed t, Equation 16 describes a continuous curve in the (k, q) plane with a kink at $(k = k^*, q = 1)$. Differentiating Equation 18 yields

$$\left.\frac{dq}{dk}\right|_{\dot{q}=0} = \frac{e^{\rho t} f''(k)}{\delta + \lambda} < 0 \qquad \text{for } q < 1. \tag{19}$$

2. The Case of $\rho = 0$

First we study the case of no technical change ($\rho = 0$). The appropriate phase diagram is given in Figure 2. Condition 14 implies that for optimality it is necessary for the following correspondence to hold:

$$\begin{aligned} s(q) &= 1 \qquad \text{when } q > 1, \\ 0 \leq s(q) &\leq 1 \qquad \text{when } q = 1, \\ s(q) &= 0 \qquad \text{when } q < 1. \end{aligned} \tag{20}$$

Then, on any given trajectory not passing through the point $(k^*, 1)$, k can be written as a continuous function of q.[4] In fact a trajectory $[k(t), q(t): t \geq 0]$

[4] By assigning the value $s(q) = 1$, the right-hand sides of differential Equations 12 and 13 are seen to be twice continuously differentiable functions of their arguments, k, q,

not passing through $(k^*, 1)$ is uniquely determined by the specification of initial conditions $[k(t_0), q(t_0); t_0]$.

Assume for purposes of exposition that the initial capital-labor ratio is the balanced capital-labor ratio k^*; that is, $k(0) = k^*$. Assume that the planning period is infinite, $T = \infty$, and that the target capital-labor ratio is left free. Then, a program of capital accumulation satisfying the necessary conditions is that of fixing $q(t) = 1$ for $0 \leq t \leq \infty$ and maintaining the balanced capital-labor ratio $k(t) = k^*$ for $0 \leq t \leq \infty$.

As δ approaches zero, this program approaches the program

$$\left(k = k^*, s = \frac{\lambda k^*}{f(k^*)}; 0 \leq t \leq \infty\right)$$

which is what Phelps [6] and Robinson [10] have dubbed the golden rule of capital accumulation. For $\delta \neq 0$, this may be called the modified golden rule of capital accumulation.[5]

If $k(0) \neq k^*$, the planning board would assign initial price q_0 such that the point (k_0, q_0) lies on a trajectory passing through $(k^*, 1)$. Let $0 \leq t^* < \infty$ be the time required for such a program to achieve $(k^*, 1)$. Then the optimal program is

$$\left(k = k^*, s = \frac{\lambda k^*}{f(k^*)}; t^* \leq t \leq \infty\right).$$

The initial savings ratio is zero or one, depending upon whether the initial capital-labor ratio is greater than or less than k^*.

The analysis is easily modified to handle the general case where $k(T) \geq$

and t, on the domain defined by $k > 0$, $q \geq 1$, $t \geq 0$. Further, by assigning the value $s(q) = 0$, the right-hand sides of Equations 12 and 13 are seen to be twice continuously differentiable functions of k, q, and t on the domain defined by $k > 0$, $q \leq 1$, $t \geq 0$. Thus, when the control $s(t)$ is appropriately assigned, the system of Equations 12 and 13 is shown to be trivially Lipschitzian over the respective domains of definition. By classic theorems of ordinary differential equations (see, e.g., pp. 159–167 in [7]), we find that for a system satisfying Equations 12 through 14 and 20, specification of the parameters $[k(t_0), q(t_0); t_0]$ uniquely determines the entire trajectory for trajectories not passing through the locus of points defined by $\{(k, q, t) \mid k = k^*(t), q = 1, t \geq 0\}$. In fact, the solutions to the system of Equations 12 through 14 vary continuously when the initial parameters $[k(t_0), q(t_0); t_0]$ are allowed to vary. See, e.g., pp. 192–199 in [7].

[5] Or, perhaps, "the adulterated golden rule." For $\rho = 0$ and $T = \infty$, it is required that $\delta > 0$ in order that the value of the definite Integral 6 be finite for all feasible programs. For $T < \infty$, the requirement that δ be positive is too strong. Even for the case with nonzero technical change, if $\delta > f'(\bar{k}_t) - \lambda$ for $t \geq 0$, then $\bar{k}_t > k_t^*$. Koopmans [3] argues that if the ethical principle is held, that all men are to be treated equally (independently of the size of their generation or its "timing"), then δ should be chosen equal to $(-n) < 0$, for the case of positive population increase. As long as $T < \infty$, our analysis is congenial to this interpretation.

$k_T \geq 0$ and $T \leq \infty$. The initial point (k_0, q_0) is chosen on a trajectory leading to the point $(k^*, 1)$, if feasibility permits. The Pontryagin program

$$\left(k = k^*, s = \frac{\lambda k^*}{f(k^*)}; t^* \leq t \leq t^{**}\right)$$

is followed where t^{**} is the time at which the backward trajectory of the system of Equations 12 through 13 starting at $(k = k_T; t = T)$ passes through $(k = k^*, q = 1)$. If, however, $q(T) < 0$ for the backward trajectory to $(k^*, 1)$ starting at k_T, then t^{**} is defined to be the time at which a backward trajectory starting at time T and demand price $q(T) = 0$ intersects the point $(k^*, 1)$. Figure 3 illustrates a program satisfying Pontryagin's necessary conditions.

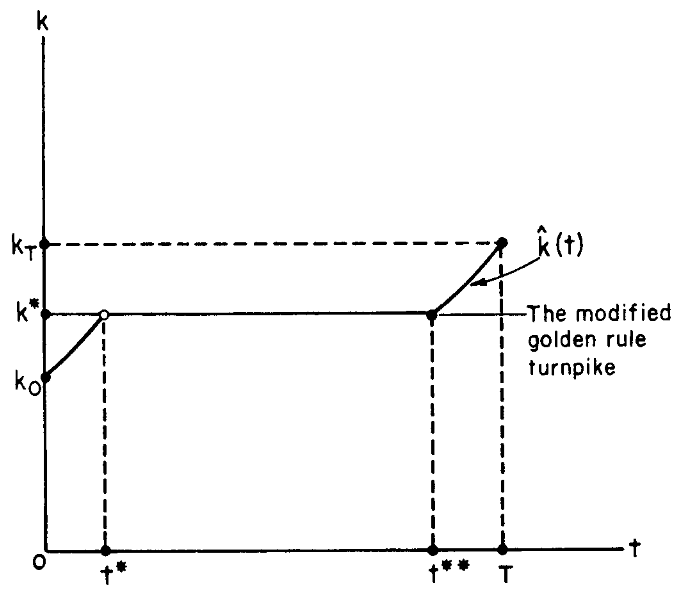

FIGURE 3. $\hat{k}(t)$, the Pontryagin path for the case $\rho = 0$.

Important assumptions are implicit in the construction of Figure 3. First, it is assumed that it is feasible for the economy with initial endowment $k(0) = k_0$ to achieve the target k_T in the specified time T. Even stronger, Figure 3 assumes that in fact

$$T > t^{**} > t^* > 0. \tag{21}$$

If it is feasible to achieve the target during the planning period but Inequality 21 fails to hold, then the Pontryagin path is the appropriate envelope of a forward trajectory from (k_0, q_0) to $(k^*, 1)$ and the backward trajectory from (k_T, q_T) to $(k^*, 1)$. In the degenerate case in which only one feasible path exists, the Pontryagin path is, of course, either a program of zero savings or a program of zero consumption. Since optimal programs do not permit the demand price of investment to become negative, if no trajectory is found with $k(T) = k_T$ and $q(T) \geq 0$, then the Pontryagin problem will yield $q(T) = 0$ and $k(T) > k_T$.

Some observations are in order. The linearity of the objective function (Integral 6) implies a kink in the graph of the stationary solutions to Equation 13. Extending the argument presented in footnote 4 on page 5, the backward solutions to the point $(k^*, 1)$ are unique. In general, however, $\hat{q}_0$ will not be uniquely determined by (k_0, k_T, T). For the degenerate Pontryagin paths that are everywhere specialized to production of the same good, there is a family of trajectories satisfying Equations 12 through 15. Nonetheless, the Pontryagin program of capital accumulation

$$[\hat{k}(t); 0 \leq t \leq T]$$

is uniquely determined by Equations 12 through 15 if a feasible program exists.

If t^* and $(T - t^{**})$ are finite this yields the following *turnpike property*: for the case of neoclassical production without technical change, the Pontryagin program of capital accumulation, if followed, requires the planning board to adopt the modified golden rule of capital accumulation for all but a finite amount of time. As the length of the planning period increases, the *fraction* of time spent on a program not satisfying the modified golden rule approaches zero.[6]

3. The Case of $\rho > 0$

Next, examine the case with positive technical progress $\rho > 0$. Notice that if ρ is nonzero, differential Equations 12 and 13 are nonautonomous, and thus the appropriate phase diagram must be drawn in three-dimensional space, (k, q, t). Time differentiation of Equation 17 yields

$$\dot{k}_t^* = \frac{-\rho(\delta + \lambda)e^{-\rho t}}{f''(k_t^*)} \gtreqless 0 \qquad \text{as } \rho \gtreqless 0. \tag{22}$$

In general, stationary solutions to the differential equation

$$\dot{q}(t) = (\delta + \lambda)q(t) - \gamma e^{\rho t} f'(k_t)$$

[6] If $\delta \leq f'(\bar{k}) - \lambda$, then the Pontryagin program $[\hat{k}(t), 0 \leq t \leq T]$ is arbitrarily close to the ratio $\bar{k}$ for all but a finite amount of time. The notion that the "turnpike property" arises in consumption-optimal programs is implicit in Cass [1], Ramsey [9], and Uzawa [16], among others, and is explicit in the recent contribution of Samuelson [11].

are shown to lie on a manifold embedded in (k, q, t) space. The manifold of solutions to $\dot{q} = 0$ is illustrated for ρ positive in Figure 4. Recalling that given t, Equation 17 has the unique solution k_t^*, suggests a program satisfying the necessary conditions of Equations 12 through 15. Consider for ease of exposition the case when the initial condition is $k(0) = k_0^*$ and the

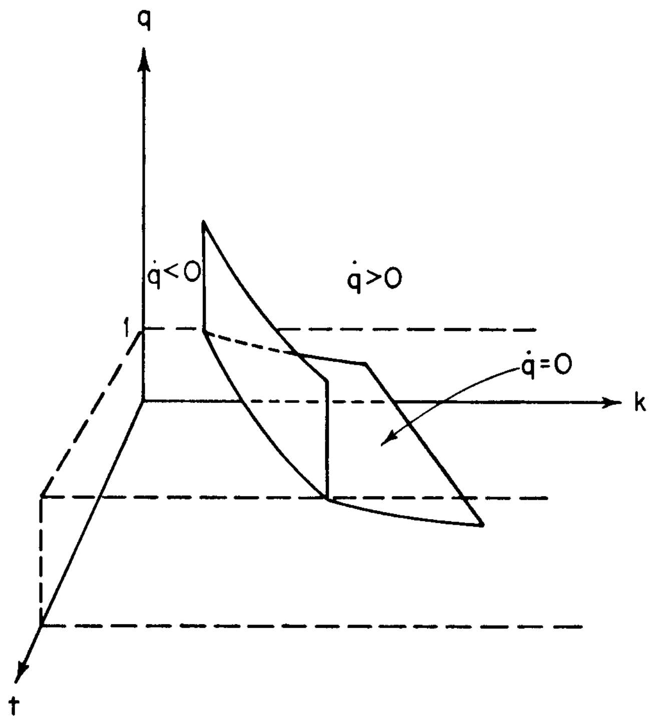

FIGURE 4. The manifold of solutions to $\dot{q} = 0$ for the case $\rho > 0$.

target requirement is $k(T) = k_T^*$. Clearly, a program of capital accumulation that follows the modified golden rule turnpike (illustrated in Figure 5) satisfies the necessary conditions of Equations 12, 13, and 15. But Equation 14 is not guaranteed. In other words, it is not guaranteed that a program of capital accumulation lying on the turnpike[7] of Figure 5 will have, for $0 \leq t \leq T$, a feasible savings ratio $0 \leq s_t \leq 1$.

[7] P. A. Samuelson has pointed out that "the turnpike" may be a misnomer for the curve $k^*(t)$ when $\rho \neq 0$. For example, in the Cobb-Douglas case with $\rho > 0$ and $s^* < 1$, if we require the target requirement to hold with equality, i.e., $k(T) = k_T$, then the fraction of time spent by the optimal program on the "turnpike" approaches $\lambda b/(\lambda b + \rho) < 1$ as T becomes large.

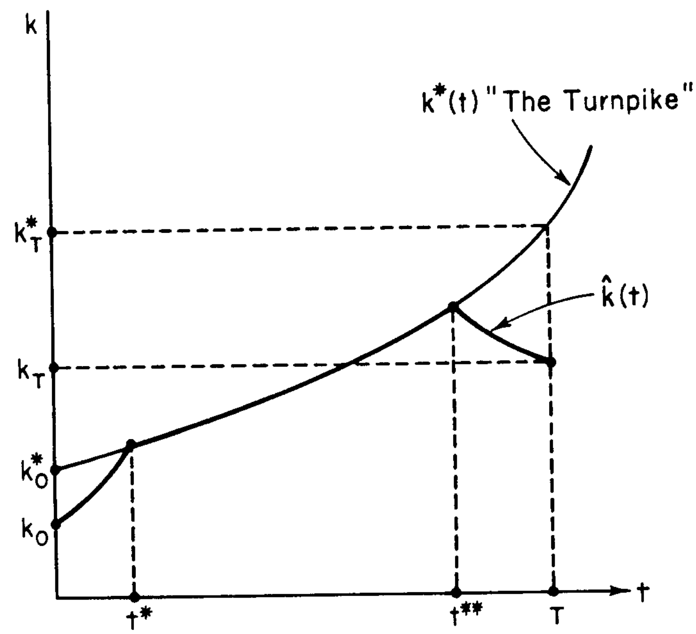

FIGURE 5. The turnpike when $\rho > 0$: the Pontryagin program of capital accumulation $\hat{k}(t)$ is shown by a heavy curve.

If $k_t = k_t^*$, Equation 12 becomes

$$\dot{k}_t = s_t e^{\rho t} f(k_t^*) - \lambda k_t^*. \tag{23}$$

The problem is to find $s_t = s_t^*$ such that, when $k_t = k_t^*$, $\dot{k} = \dot{k}_t^*$. Equating $\dot{k}_t$ to $\dot{k}_t^*$ yields

$$s^* e^{\rho t} f(k_t^*) - \lambda k_t^* = \frac{-\rho(\delta + \lambda)e^{-\rho t}}{f''(k_t^*)}$$

from Equations 22 and 23. Or rewriting

$$s_t^* = \frac{\lambda k_t^*}{e^{\rho t} f(k_t^*)} - \frac{\rho(\delta + \lambda)}{e^{2\rho t} f(k_t^*) f''(k_t^*)} > 0 \qquad \text{for } \rho > 0. \tag{24}$$

This is the common-sense result: to achieve a program of positive capital accumulation requires a positive savings fraction. However, Equation 24 does not guarantee $s_t^* \leq 1$ for $\rho > 0$. To see this, consider the case where the production function is linear-logarithmic in capital and labor, $y_t = e^{\rho t} k_t^a$. Let $0 < a < 1$ so that a is capital's share of output in a competitive economy and the production function is Cobb-Douglas. For the Cobb-Douglas case

$$k_t^* = \left(\frac{a e^{\rho t}}{\delta + \lambda}\right)^{1/b}$$

and

$$k_t^* = \frac{\rho}{b}\left(\frac{ae^{\rho t}}{\delta + \lambda}\right)^{1/b},$$

where b is defined by $b = 1 - a$. Therefore, for the Cobb-Douglas case,

$$s^* = \frac{a(\lambda b + \rho)}{b(\lambda + \delta)}.$$

For this special case, s^* is independent of time and greater than zero; but whether s^* is less than, equal to, or greater than unity depends upon the values of the parameters a, ρ, λ, δ.[8]

Returning to the case of general neoclassical production, an example of a Pontryagin program of capital accumulation is presented in Figure 5. In drawing this figure it is implicitly assumed that $0 < t^* < t^{**} < T$ and therefore that $s^*(t) \leq 1$ for $t^* < t < t^{**}$. It is further assumed that $\hat{q}(T) \geq 0$ when $\hat{k}(T) = k_T$.

The general case where s^* changes with time presents a sophisticated mathematical difficulty. If the number of switches from $s^* < 1$ to $s^* > 1$, and vice versa, is sufficiently large, it may be impossible to find a *piecewise continuous* control $\hat{s}(t)$ satisfying Equations 12 through 15. If no such control exists, then no maximum to Integral 6 exists.[9]

4. Optimality of the Pontryagin Program

In the previous sections, programs satisfying necessary Conditions 7–10 and 12–15 are referred to as Pontryagin programs. It remains to show that the necessary conditions are also sufficient, that such programs are indeed optimal.[10]

[8] Consider the "familiar economy" where $a = .30$, $\lambda = n + \mu = .10$, and $\rho = .03$. If the planning board's rate of discount $\delta = .05$, then $s^* = \frac{2}{7} < 1$. Hence if the "familiar economy" achieves the capital-labor ratio $k^*(t)$, at time t, then it can maintain the "turnpike" capital-labor ratio. It is not surprising that s^* is independent of t for Cobb-Douglas functions. Since technical change is labor augmenting in this case, to remain on the turnpike it is required that the capital-labor ratio *measured in efficiency units* be held constant. Indeed, if the parameters δ and λ are replaced by $\delta^0 = \delta + (\rho/b)$ and $\lambda^0 = \lambda + (\rho/b)$, respectively, then the analysis follows that of section 2. For example, the feasibility condition $s^* < 1$ simply reduces to the condition $\delta^0 > f'(\hat{k}) - \lambda^0$.

[9] If $T < \infty$ and the class of admissible controls $[0 \leq s(t) \leq 1; 0 \leq t \leq T]$ is restricted to piecewise continuous functions, then a maximum to Integral 6 exists if and only if the number of such switches in $[0, T]$ is finite. Therefore if s^* is an analytic function of t, then a maximum to Integral 6 exists.

[10] It is essential to impose some measurability requirement upon the set of admissible controls $[0 \leq s(t) \leq 1; 0 \leq t \leq T]$. If, as implied by Equation 14, attention is restricted to those controls that are piecewise continuous, then the integration performed in Expressions 6 and 25 through 30 is to be interpreted in the sense of Stieltjes. On the other hand, if attention is restricted to Lebesgue measurable controls, then the integration in Expressions 6 and 25 through 30 is to be interpreted in the sense of Lebesgue.

Let $[\hat{c}(t), \hat{z}(t), \hat{k}(t), \hat{q}(t), \cdots]$ be a program satisfying the conditions of Equations 7 through 10 and 12 through 15. Let $[c(t), z(t), k(t), q(t), \cdots]$ be any feasible program, that is, any program satisfying Conditions 7 through 10. It is necessary to show

$$\int_0^T (\hat{c} - c)e^{-\delta t}\, dt \geq 0. \tag{25}$$

The left-hand side of Inequality 25 can be rewritten in the form

$$\int_0^T e^{-\delta t}\, dt\{(\hat{c} - c) + \hat{\gamma}[(e^{\rho t}f(\hat{k}) - \hat{z} - \hat{c}) - (e^{\rho t}f(k) - z - c)] + \hat{q}[(\hat{z} - \lambda\hat{k} - \dot{\hat{k}}) - (z - \lambda k - \dot{k})]\},$$

which reduces to

$$\int_0^T e^{-\delta t}\, dt\{(1 - \hat{\gamma})(\hat{c} - c) + (\hat{q} - \hat{\gamma})(\hat{z} - z) + \hat{\gamma}e^{\rho t}[f(\hat{k}) - f(k)] + \hat{q}[\lambda(k - \hat{k}) + (\dot{k} - \dot{\hat{k}})]\}. \tag{26}$$

Notice that

$$(1 - \hat{\gamma})(\hat{c} - c) \geq 0$$

and

$$(\hat{q} - \hat{\gamma})(\hat{z} - z) \geq 0.$$

Therefore Integral 26 is not less than the following expression

$$\int_0^T e^{-\delta t}\, dt\{\hat{\gamma}e^{\rho t}[f(\hat{k}) - f(k)] + \hat{q}[\lambda(k - \hat{k}) + (\dot{k} - \dot{\hat{k}})]\}. \tag{27}$$

But since $f(\cdot)$ is a concave function, Integral 27 is not smaller than

$$\int_0^T e^{-\delta t}\, dt\{\hat{\gamma}e^{\rho t}[(\hat{k} - k)f'(\hat{k})] + \hat{q}[\lambda(k - \hat{k}) + (\dot{k} - \dot{\hat{k}})]\}.$$

By collecting terms the previous expression yields

$$\int_0^T \hat{q}e^{-\delta t}(\dot{k} - \dot{\hat{k}})\, dt + \int_0^T e^{-\delta t}\, dt(\hat{k} - k)[\hat{\gamma}e^{\rho t}f'(\hat{k}) - \hat{q}\lambda]. \tag{28}$$

Integrating the first term in Expression 28 by parts reduces to

$$\hat{q}(T)e^{-\delta T}[k(T) - \hat{k}(T)] - \hat{q}_0[k(0) - \hat{k}(0)] - \int_0^T (k - \hat{k})(\dot{\hat{q}} - \delta\hat{q})e^{-\delta t}\, dt. \tag{29}$$

The transversality condition of Equation 15 says that the first term in Expression 29 is nonnegative. Since every feasible path must satisfy the given initial condition k_0, the second term in Expression 29 is identically zero. Hence

$$\int_0^T \hat{q}e^{-\delta t}(\dot{k} - \dot{\hat{k}})\, dt \geq -\int_0^T (k - \hat{k})(\dot{\hat{q}} - \delta\hat{q})e^{-\delta t}\, dt. \tag{30}$$

Hence Integral 28 is not smaller than

$$\int_0^T e^{-\delta t}\, dt\{(\hat{k} - k)[\hat{\gamma} e^{\rho t} f'(\hat{k}) - \lambda \hat{q}] + (\hat{k} - k)(\dot{\hat{q}} - \delta \hat{q})\}$$
$$= \int_0^T e^{-\delta t}\, dt(\hat{k} - k)[\dot{\hat{q}} - (\delta + \lambda)\hat{q} + \hat{\gamma} e^{\rho t} f'(\hat{k})],$$

which by Conditions 13 and 14 is identically zero. Hence the optimality requirement of Inequality 25 is established. In fact, if $k \neq \hat{k}$ on some interval, then Inequality 25 is strict.

5. The Two-Sector Model

Consider the two-sector model of economic growth that was introduced by Meade [4] and Uzawa [15]. The model economy consists of an investment-goods sector and a consumption-goods sector, labeled 1 and 2, respectively. In both sectors, production is subject to constant returns to scale, and marginal rates of substitution are positive. There are no external economies (diseconomies) and no joint products.

The quantity of the consumption goods $Y_2(t)$ currently produced depends upon the current allocation $K_2(t)$ and $L_2(t)$ of capital and labor to the consumption sector:

$$Y_2(t) = M(t)F_2[K_2(t), L_2(t)]. \tag{31}$$

Similarly, current production of the investment goods $Y_1(t)$ is dependent upon the current allocation of factors to the investment sector:

$$Y_1(t) = G(t)F_1[K_1(t), L_1(t)]. \tag{32}$$

We have $F_1[\cdot]$ and $F_2[\cdot]$ as neoclassical production functions, homogeneous of degree one in their respective arguments. Thus

$$F_j(\lambda K_j, \lambda L_j) = \lambda F_j(K_j, L_j) \qquad \text{for } j = 1, 2, \tag{33}$$

where $K_j, L_j \geq 0$ and $\lambda > 0$. Labor and capital can be freely shifted between the two sectors.[11] For an allocation of resources to be feasible at time t,

$$\begin{aligned} K_1(t) + K_2(t) &\leq K(t), \\ L_1(t) + L_2(t) &\leq L(t), \end{aligned} \tag{34}$$

with $K_1(t)$, $K_2(t)$, $L_1(t)$, $L_2(t) \geq 0$, where $K(t) > 0$ and $L(t) > 0$ are the current stocks of available capital and labor.

If the capital stock is subject to evaporative decay at the constant rate $\mu > 0$, then growth of the capital stock is specified by the differential equation

$$\dot{K}(t) = Y_1(t) - \mu K(t). \tag{35}$$

[11] In the terminology of Meade [4], the factors of production are assumed to be perfectly malleable.

Assume that labor is inelastically supplied, that it is a stationary fraction of total population, and grows at the constant relative rate n,

$$\dot{L}(t) = nL(t). \tag{36}$$

In Equations 31 and 32 it is assumed that technical change in the two sectors is of the Hicks-neutral type. Assume further that change in the respective levels of technique is independent of other economic variables and proceeds at constant relative rates,

$$\dot{G}(t) = gG(t) \quad \text{and} \quad \dot{M}(t) = mM(t). \tag{37}$$

Formally the problem is to maximize

$$\int_0^T \frac{C(t)}{L(t)} e^{-\delta t}\, dt, \tag{38}$$

where δ is the social rate of time discount and T is the length of the planning period. The maximand of Expression 38 is constrained by the Conditions 31 through 37 and by the given initial conditions $K(0) = K_0$, $L(0) = L_0$, $G(0) = G_0$, $M(0) = M_0$, and subject to the requirement that the terminal capital-labor ratio be at least as great as some specified target, $k(T) \geq k_T$.

In order to facilitate the exposition, certain important constructions introduced in [15], [16], and [5] are reproduced in this section. Define the per capita quantities:

$$y_j(t) = \frac{Y_j(t)}{L(t)},$$

$$k_j(t) = \frac{K_j(t)}{L_j(t)},$$

$$l_j(t) = \frac{L_j(t)}{L(t)},$$

$$f_j(k_j) = F_j(k_j, 1) \qquad \text{for } j = 1, 2,$$

$$k(t) = \frac{K(t)}{L(t)}.$$

Assume that $f_j(k_j)$ is three times continuously differentiable and

$$\begin{aligned} &f_j(k_j) > 0, \quad f_j'(k_j) > 0, \quad f_j''(k_j) < 0 \qquad \text{for } 0 < k_j < \infty, \\ &\lim_{k_j \to 0} f_j(k_j) = 0, \quad \lim_{k_j \to \infty} f_j(k_j) = \infty, \\ &\lim_{k_j \to 0} f_j'(k_j) = \infty, \quad \lim_{k_j \to \infty} f_j'(k_j) = 0. \end{aligned} \tag{39}$$

Because positive marginal products are assumed, optimality requires that Inequalities 34 hold with equality. If ω is an arbitrarily given wage-rentals ratio, then efficient capital-labor ratios can be found by solving for k_j,

$$\omega = \frac{f_j(k_j)}{f_j'(k_j)} - k_j \qquad \text{for } j = 1, 2. \tag{40}$$

From Conditions 39 and 40,

$$\frac{dk_j}{d\omega} = \frac{-[f_j'(k_j)]^2}{f_j(k_j)f_j''(k_j)} > 0. \tag{41}$$

Thus the efficient capital-labor ratio k_j is a uniquely determined, increasing function of the wage-rentals ratio ω.

Define $p(\omega, t)$ the supply price of a unit of investment goods at time t,

$$p(\omega, t) = \frac{M_0 e^{mt} f_2'[k_2(\omega)]}{G_0 e^{gt} f_1'[k_1(\omega)]}, \tag{42}$$

where a unit of consumption goods is the *numéraire*. Logarithmic differentiation of Equation 42 yields

$$\frac{1}{p}\frac{\partial p}{\partial \omega} = \frac{1}{k_1(\omega) + \omega} - \frac{1}{k_2(\omega) + \omega} \gtrless 0 \qquad \text{as } k_2 \gtrless k_1. \tag{43}$$

The Conditions 34 and 39 imply that

$$k_1 l_1 + k_2 l_2 = k, \tag{44}$$

$$l_1 + l_2 = 1, \tag{45}$$

where $k_1, k_2, l_1, l_2 \geq 0$. Given k and t, Equations 44 and 45, together with Equations 40 and 42, define the range of p and ω. This is illustrated graphically for the case $k_2(\omega) > k_1(\omega)$ in Figure 6. In general, define the critical wage-rentals ratios by

$$\begin{aligned} \omega_{\min}(k) &= \min\,[\omega_2(k), \omega_1(k)], \\ \omega_{\max}(k) &= \max\,[\omega_2(k), \omega_1(k)], \end{aligned} \tag{46}$$

and the critical supply prices by

$$\begin{aligned} p_{\min}(k, t) &= p_2(k, t), \\ p_{\max}(k, t) &= p_1(k, t), \end{aligned} \tag{47}$$

where $k_j(\omega_j) = k$ and $p_j(k, t) = p[\omega_j(k), t]$ for $j = 1, 2$.

It is possible to choose units of measurement such that $G_0 = M_0 = 1$. From Equations 31, 32, 44, and 45, we have

$$y_1 = \frac{k_2 - k}{k_2 - k_1} e^{gt} f_1(k_1), \tag{48}$$

$$y_2 = \frac{k - k_1}{k_2 - k_1} e^{mt} f_2(k_2), \tag{49}$$

where the variables y_1, y_2, k, k_1, k_2 are understood to be functions of time.

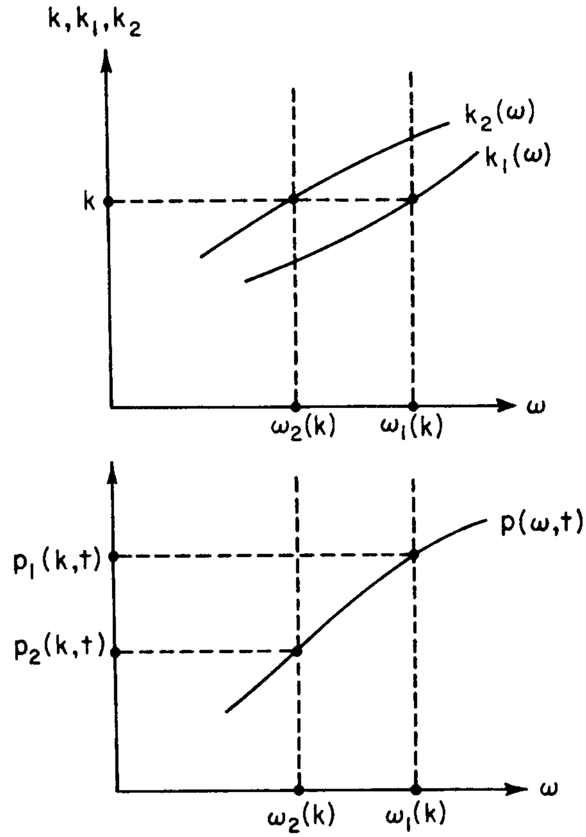

FIGURE 6. Determination of critical wage-rentals ratios and critical price ratios.

Thus given k, ω, and t, Equations 40, 42, 48, and 49 uniquely determine the allocation of factors between the two sectors, the level of production in both sectors,[12] and the implicit price of a unit of the investment good in terms of a unit of the consumption good. Partial differentiation of Equations 48 and 49 yields

$$\frac{\partial y_1}{\partial k} = \frac{-e^{\sigma t} f_1(k_1)}{k_2 - k_1}, \tag{50}$$

$$\frac{\partial y_2}{\partial k} = \frac{e^{mt} f_2(k_2)}{k_2 - k_1}, \tag{51}$$

[12] As long as $k_2(\omega) \neq k_1(\omega)$.

$$\frac{\partial y_1}{\partial \omega} = e^{gt}\left[\left(\frac{k_2 - k}{(k_2 - k_1)^2}\right)\left(\frac{f_1'^2}{f_1 f_1''}\left[-k_2 f_1' - (f_1 - k_1 f_1')\right]\right) + \left(\frac{k_1 - k}{(k_2 - k_1)^2}\right)\left(\frac{f_2'^2 f_1}{f_2 f_2''}\right)\right], \tag{52}$$

$$\frac{\partial y_2}{\partial \omega} = e^{mt}\left[\left(\frac{k - k_1}{(k_2 - k_1)^2}\right)\left(\frac{f_2'^2}{f_2 f_2''}\left[k_1 f_2' + (f_2 - k_2 f_2')\right]\right) + \left(\frac{k_2 - k}{(k_2 - k_1)^2}\right)\left(\frac{f_1'^2 f_2}{f_1 f_1''}\right)\right]. \tag{53}$$

The implicit value of gross national product per worker y is defined by

$$y = y_2 + py_1. \tag{54}$$

It is useful to define s, the fraction of implicit gross national product assigned to investment,

$$s = \frac{py_1}{y}. \tag{55}$$

6. Optimal Control in the Two-Sector Model

Maximization of Integral 38 is equivalent to maximization of

$$\int_0^T y_2 e^{-\delta t}\, dt. \tag{56}$$

Without loss in generality, the central planning board can consider $s(t)$ to be the control variable chosen from among all, say, piecewise continuous functions defined upon $0 \leq t \leq T$, such that $0 \leq s(t) \leq 1$ for $0 \leq t \leq T$.

The problem reduces to choosing $s(t)$ to maximize

$$\int_0^T [1 - s(t)]y(t)e^{-\delta t}\, dt, \tag{57}$$

subject to the constraints:

$$\dot{k}(t) = \frac{s(t)y(t)}{p(t)} - \lambda k \qquad \text{where } \lambda = \mu + n, \tag{58}$$

$$0 \leq s(t) \leq 1 \qquad \text{for } 0 \leq t \leq T \tag{59}$$

and $s(t)$ is a piecewise continuous function of t,

$$k(0) = k_0 \quad \text{and} \quad k(T) \geq k_T. \tag{60}$$

As stated, this problem is soluble as an application of Pontryagin's maximum principle. First form the Hamiltonian expression

$$H(k, q, s, t) = (1 - s)e^{-\delta t}y + qe^{-\delta t}\left(\frac{sy}{p} - \lambda k\right), \tag{61}$$

where $q(t)$ is the current social demand price of a unit of the investment good in terms of a unit of the consumption good. From Equations 40, 42, 48, 49, and 54, y and p are interpreted as functions of k, s, and t.

It is necessary for optimality that $s(t)$ be chosen in [0, 1] to maximize the socially imputed value of gross national product at time t,

$$(1 - s)y + q\left(\frac{sy}{p}\right). \tag{62}$$

Thus it is necessary for optimality that

$$s = 1, \quad k_1 = k, \quad l_1 = 1; \tag{63}$$

or

$$s = 0, \quad k_2 = k, \quad l_2 = 1; \tag{64}$$

or

$$\frac{\partial(1 - s)y}{\partial s} + q\frac{\partial(sy/p)}{\partial s} = 0. \tag{65}$$

Observe that by employing Equations 48 and 49, Equation 65 can be rewritten as

$$\frac{\partial y_1}{\partial \omega}\frac{\partial \omega}{\partial s} = -q\frac{\partial y_2}{\partial \omega}\frac{\partial \omega}{\partial s}. \tag{66}$$

But from Equations 40, 42, 52, and 53,

$$\frac{\partial y_2/\partial \omega}{\partial y_1/\partial \omega} = -p. \tag{67}$$

Hence if the maximum to Expression 62 is interior and thus characterized by Equation 65, then with $q(t)$ given, s is chosen such that $p(k, s, t) = q(t)$. Similarly if Equations 63 apply, then s has been chosen such that $p(k, s, t) < q(t)$; if Equations 64 apply then s has been chosen such that $p(k, s, t) > q(t)$.

Pontryagin's second necessary condition is that the social demand price of investment change through time in a manner reflecting the planning board's perfect foresight of the imputed marginal value product of capital,

$$\dot{q} = (\delta + \lambda)q - \left[\frac{\partial(1 - s)y}{\partial k} + q\frac{\partial(sy/p)}{\partial k}\right]. \tag{68}$$

From Equations 40 through 42, 50 through 53, and 67, we have

$$\frac{\partial(sy/p)}{\partial k} = \frac{\partial y_1}{\partial k} + \frac{\partial y_1}{\partial \omega}\frac{\partial \omega}{\partial k} = -e^{gt}f_1'\left(\frac{k_1 + \omega}{k_2 - k_1}\right) + \frac{\partial y_1}{\partial \omega}\frac{\partial \omega}{\partial k}, \tag{69}$$

and

$$\frac{\partial(1 - s)y}{\partial k} = \frac{\partial y_2}{\partial k} + \frac{\partial y_2}{\partial \omega}\frac{\partial \omega}{\partial k} = p\left[e^{gt}f_1'\left(\frac{k_2 + \omega}{k_2 - k_1}\right) - \frac{\partial y_1}{\partial \omega}\frac{\partial \omega}{\partial k}\right]. \tag{70}$$

Therefore if Equation 65 applies, and hence $p(k, s, t) = q(t)$, then differential Equation 68 reduces to

$$\dot{q} = [\delta + \lambda - e^{gt}f_1'(k_1)]q. \quad (71)$$

Also observe that if Equations 63 apply, that is, $p(k, s, t) < q(t)$, then Equation 68 reduces to Equation 71. But if Equations 64 apply, that is, $p(k, s, t) > q(t)$, then Equation 68 reduces to

$$\dot{q} = (\delta + \lambda)q - e^{mt}f_2'(k). \quad (72)$$

7. The Case where Production of the Consumption Good is more Capital Intensive than Production of the Investment Good

It is convenient to treat the general problem posed in section 6 by separate cases depending upon certain attributes of the techniques of production implied by Equations 31 and 32. This section treats the case where Equations 31 and 32 are such that

$$k_2(\omega) > k_1(\omega) \qquad \text{for } \omega > 0. \quad (73)$$

It is necessary to study the behavior of the nonautonomous pair of differential Equations 58 and 68. Equations 47 divide the (k, q, t)-phase space into three mutually exclusive regions:

$$S_1 = \{(k, q, t) \mid q > p_{\max}(k, t)\}, \quad (74)$$

$$S_2 = \{(k, q, t) \mid q < p_{\min}(k, t)\}, \quad (75)$$

$$N = \{(k, q, t) \mid p_{\min}(k, t) \leq q \leq p_{\max}(k, t)\}. \quad (76)$$

In region S_1, maximization of the imputed value of gross national product of Expression 62 implies specialization to the production of the investment good. Therefore Equations 58 and 68 reduce to

$$\dot{k} = e^{gt}f_1(k) - \lambda k, \qquad \text{in } S_1, \quad (77)$$

and

$$\dot{q} = [(\delta + \lambda) - e^{gt}f_1'(k)]q \qquad \text{in } S_1. \quad (78)$$

Likewise, maximization of Expression 62 requires that in region S_2 the economy be specialized to the production of the consumption good. Thus Equations 58 and 68 reduce to

$$\dot{k} = -\lambda k \qquad \text{in } S_2, \quad (79)$$

and

$$\dot{q} = (\delta + \lambda)q - e^{mt}f_2'(k) \qquad \text{in } S_2. \quad (80)$$

In the region N (for nonspecialization), maximization of Expression 62 implies Equation 65 and therefore by Equations 66, 67, 69, and 70, Equations 58 and 68 reduce to

$$\dot{k} = e^{gt} f_1[k_1(\omega)] \frac{k_2(\omega) - k}{k_2(\omega) - k_1(\omega)} - \lambda k \qquad \text{in } N, \tag{81}$$

and

$$\dot{q} = \{(\delta + \lambda) - e^{gt} f_1'[k_1(\omega)]\} q \qquad \text{in } N, \tag{82}$$

where

$$q(t) = p(\omega, t) \qquad \text{in } N. \tag{83}$$

By assumption of Inequality 73 and Equations 43, p is a strictly increasing function of ω. Therefore specification of $q(t)$ and t uniquely determines ω, which in turn uniquely determines $k_1(\omega)$ and $k_2(\omega)$ by Equations 40 and 41. Thus by Equations 48 and 49 the right-hand sides of Equations 81 and 82 are uniquely determined by specification of $(k, q, t) \in N$.

The problem is to characterize the behavior of the system in Equations 77 through 83 in (k, q, t)-space. Notice that by Relations 43, 47, and 73, $p_{\max}(k, t)$ and $p_{\min}(k, t)$ are strictly increasing functions of k. From Equation 42,

$$\frac{\partial p(\omega, t)}{\partial t} = \frac{(m - g)e^{(m-g)t} f_2'[k_2(\omega)]}{f_1'[k_1(\omega)]}$$

and therefore

$$\operatorname{sgn}\left(\frac{\partial p}{\partial t}\right) = \operatorname{sgn}(m - g). \tag{84}$$

In particular,

$$\operatorname{sgn}\left(\frac{\partial p_{\max}}{\partial t}\right) = \operatorname{sgn}\left(\frac{\partial p_{\min}}{\partial t}\right) = \operatorname{sgn}(m - g).$$

It follows from Equation 78, that for $(k, q, t) \in S_1$, $\dot{q} = 0$ if and only if

$$e^{gt} f_1'(k) = \delta + \lambda. \tag{85}$$

Conditions 39, together with the requirement that $\delta > 0$ and $\lambda > 0$, ensure that there exists a uniquely determined function of t, $k_1^*(t)$ that solves Equation 85. In fact, time differentiation of Equation 85 yields

$$\frac{dk_1^*(t)}{dt} = \frac{-ge^{-gt}}{f_1''[k_1^*(t)]},$$

Applying Conditions 39 yields

$$\operatorname{sgn}\left(\frac{dk_1^*}{dt}\right) = \operatorname{sgn}(g). \tag{86}$$

Further define

$$q_t^* = p_{\max}[k_1^*(t), t]. \tag{87}$$

Since $p_{\max}$ is an increasing function of k, Equation 87 is well defined. Further, consider the equation

$$q_t^* = p(\omega, t). \tag{88}$$

Given t, Equation 88 has a unique solution $\omega = \omega_t^*$. Since $k_1^* = k_1(\omega_t^*, t)$, from Equation 82 for $(k, q, t) \in N$, $\dot{q} = 0$ if and only if $q(t) = q_t^*$.

For $(k, q, t) \in S_2$, $\dot{q} = 0$ if and only if

$$q_t = \frac{e^{mt} f_2'(k_t)}{\delta + \lambda} \tag{89}$$

by Equation 80. Equation 89 gives q_t as a function of k_t and t with

$$\left(\frac{\partial q_t}{\partial k}\right)_{\dot{q}=0} = \frac{f_2''(k_t)}{\delta + \lambda} < 0 \qquad \text{for } (k, q, t) \in S_2.$$

Next it is required to describe the set of points that yield stationary solutions to the capital accumulation Equation 58. For $(k, q, t) \in S_1$, $\dot{k} = 0$ if and only if

$$e^{gt} f_1(k_t) = \lambda k_t. \tag{90}$$

Conditions 39 ensure that the solution to Equation 90, $\bar{k}_t$, is a well-defined function of t. If there is no technical change in the production of the investment good, that is, $g = 0$, then $\bar{k}$ has the interpretation of the maximum sustainable capital-labor ratio. Also when $\lambda > 0$ and $\delta \geq 0$, $\bar{k}_t > k_1^*(t)$ for all t. Time differentiation of Equation 90 yields

$$\dot{\bar{k}}_t = g\left[\frac{\lambda \bar{k}_t}{\lambda - e^{gt} f_1'(\bar{k}_t)}\right].$$

From Conditions 39, the average product of capital is always greater than the marginal product of capital, and thus

$$\operatorname{sgn}(\dot{\bar{k}}_t) = \operatorname{sgn}(g).$$

For $(k, q, t) \in N$ and $k_t > \bar{k}_t$, there are no stationary solutions to the capital accumulation Equation 81. However, for $(k, q, t) \in N$ and $k_t < \bar{k}_t$ stationary solutions to Equation 81 are such that $p_{\min}(k, t) < q_t < p_{\max}(k, t)$. Of course, for $(k, q, t) \in S_2$, there are no stationary solutions to the capital accumulation Equation 79 with $k_t > 0$.

For purposes of exposition, first consider the case of no technical change, $m = g = 0$. For this special case, the system of differential Equations 58 and 68 is autonomous and thus can be characterized by the two-dimensional

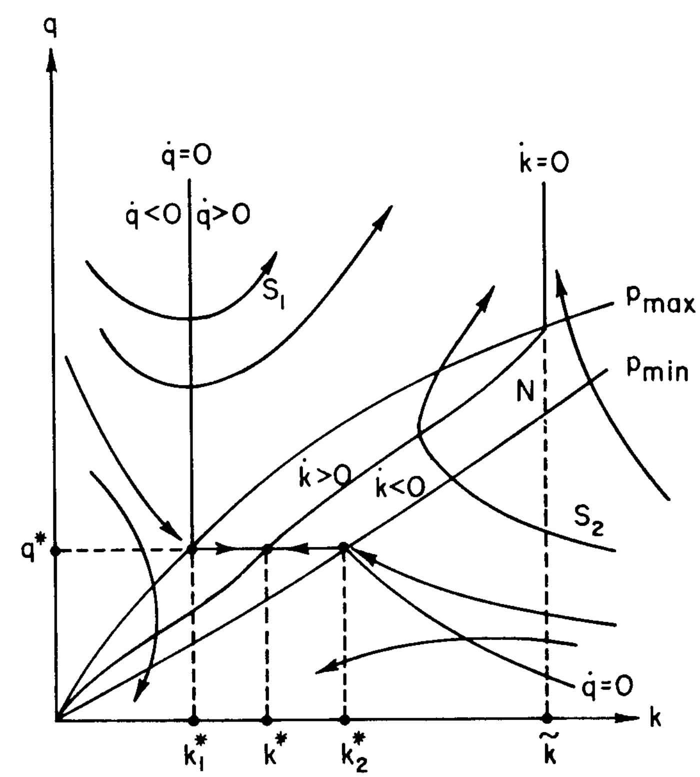

FIGURE 7. Phase diagram fo- the case where consumption goods production is more capital intensive and where there is no technical change.

phase diagram of Figure 7. The intersection of the locus $\dot{q} = 0$ with the locus $\dot{k} = 0$ is shown to be the point (k^*, q^*). To verify that (k^*, q^*) is a saddle point for the system of Equations 77 through 83, consider the linear Taylor approximations to Eq''ations 81 and 82 evaluated at (k^*, q^*). The roots to the relevant characteristic equation are

$$\frac{1}{2}\left[\frac{\partial \dot{k}}{\partial k} + \frac{\partial \dot{q}}{\partial q} \pm \sqrt{\left(\frac{\partial \dot{k}}{\partial k} + \frac{\partial \dot{q}}{\partial q}\right)^2 - 4\,\frac{\partial \dot{k}}{\partial k}\frac{\partial \dot{q}}{\partial q}}\right]_{(k^*,q^*)}.$$

But $(k^*, q^*) \in N$, thus

$$\left.\frac{\partial \dot{q}}{\partial q}\right|_{(k^*,q^*)} = \left[-qf_1''(k_1)\frac{dk_1}{d\omega}\frac{\partial \omega}{\partial q}\right]_{(k^*,q^*)} > 0,$$

and

$$\frac{\partial \dot{k}}{\partial k} = \frac{-f_1(k_1)}{k_2 - k_1} - \lambda < 0,$$

since $k_2 > k_1$. The characteristic roots are real and opposite in sign and therefore the unique singular point (k^*, q^*) is a saddle point.[13]

If the length of the planning period is infinite, $T = \infty$, and the "terminal" capital-labor ratio $k(\infty)$ is allowed to be free, the optimal program of capital accumulation is such that $\lim_{t\to\infty} k(t) = k^*$. If $0 < k_0 < k_1^*$, then $\hat{q}(0)$ is chosen such that $[k_0, \hat{q}(0)]$ is on the unique backward solution from the point (k_1^*, q^*). Thus, for $0 \leq t < t^*$, the savings fraction $\hat{s}(t) = 1$, where t^* is defined by

$$t^* = \int_{k_0}^{k_1^*} \frac{dk}{f_1(k) - \lambda k}.$$

For $t^* \leq t \leq \infty$, $\hat{\omega} = \omega^*$, which determines $\hat{k}_1 = k_1^*$ and $\hat{k}_2 = k_2^*$. Since $\lim_{t\to\infty} \hat{q}(t) = q^*$, the transversality condition is seen to hold.[14]

For $k_0 > k_2^*$, $q(0)$ is chosen such that $[k_0, q(0)]$ lies on the unique backward solution going through the point (k_2^*, q^*). For $0 \leq t < t^{**}$, the optimal savings fraction $\hat{s}(t) = 0$, where t^{**} is defined by

$$t^{**} = \frac{\log (k_0/k_2^*)}{\lambda}.$$

For $t^{**} \leq t \leq \infty$, set $\hat{\omega} = \omega^*$ and thus $\hat{k}_1 = k_1^*$ and $\hat{k}_2 = k_2^*$. As in the previous case, the transversality conditions are seen to hold on such a path.

For the more general case when the planning period may be finite, $T \leq \infty$, the optimal path (if feasible) is determined by specification of the vector (k_0, k_T, T). The program of capital accumulation thus determined is uniquely determined.[15] As in the one-sector case, the optimal paths of capital accumulation possess a certain "turnpike" property. Heuristically, it can be ascertained from Figure 7 that with (k_0, k_T) fixed, as $T \to \infty$, the optimal capital-labor ratio is arbitrarily close to the balanced capital-labor ratio k^* for all but a finite amount of time.[16]

Next, we extend these results to include the case of nonzero technical change. In order to guarantee that the value of the definite Integral 56 is finite on all feasible paths, we shall restrict our attention in what follows to

[13] Cf. pp. 246–254 in [7].

[14] Theorem 3, p. 50, in [8] requires for constrained optimality of Integral 56 that $e^{-\delta T} q(T)[k(T) - k_T] = 0$. That is, for optimality it is required that either the terminal target (Condition 60) hold with equality or that the terminal social demand price of investment be zero.

[15] See pp. 159–167 and pp. 192–199 in [7].

[16] In the one-sector model, the optimal program is such that the optimal capital-labor ratio is *equal* to the balanced capital-labor ratio for all but a finite amount of time. In the one-sector case, the production possibility set is an isosceles triangle. Thus $q(t)$ can be varied continuously while the optimal savings ratio jumps from one or zero to the balanced savings ratio. However, if $k_1(\omega) \neq k_2(\omega)$, then the production possibility frontier is *strictly* concave.

cases in which the length of the planning period is finite, $0 < T < \infty$. In general, the system of differential Equations 77 through 83 is nonautonomous and thus is appropriately characterized in the (k, q, t)-phase space. Since such figures are difficult to represent graphically, we shall present instead (k, q) "snapshots" of the basic (k, q, t)-phase space.

Consider, for example, the case where there is technical progress in both sectors, but progress in the production of the consumption good is more rapid than progress in the production of the investment good; that is,

$$m > g > 0. \tag{91}$$

From Equations 84, 86 through 90, and Inequality 91, we have

$$\begin{gathered} \frac{dk_1^*(t)}{dt} > 0, \\ \frac{\partial p_{\max}(k, t)}{\partial t} > 0, \quad \frac{\partial p_{\min}(k, t)}{\partial t} > 0, \\ \frac{d\bar{k}(t)}{dt} > 0, \quad \text{and thus} \\ \frac{dq^*(t)}{dt} > 0 \qquad \text{for } m > g > 0. \end{gathered} \tag{92}$$

A "snapshot" of the relevant phase diagram for the case of Inequality 91 is given in Figure 8. The schedules $p_{\max}(k)$ and $p_{\min}(k)$ are shown for time t, thus dividing the space into the three regions S_1, S_2, and N. The region N is shown cross-hatched. The loci of points that satisfy $\dot{q} = 0$ and $\dot{k} = 0$ at time t are indicated by heavy solid lines. The direction of shifts of the respective loci as t increases are indicated by the heavy dashed arrows. Thus, for example, as t increases, the straight line q_t^* generates a surface on which $dq^*(t)/dt$ is positive for all time $t \geq 0$.

The reader should use Figure 8 as an aid in visualizing the full three-dimensional phase diagram. Specification of the three boundary conditions (k_0, k_T, T) determines, if feasible, a path that satisfies Equations 77 through 83. The program of capital accumulation $\hat{k}(t)$ for $0 \leq t \leq T$, is thus uniquely determined. If on the path just chosen, the terminal value of the social demand price of investment is negative $q(T) < 0$, choose instead the path (yielding higher welfare) that is determined by the three boundary conditions $[k(0) = k_0, q(T) = 0, T]$.

The case where technical progress in the production of the investment good proceeds at a greater rate than technical progress in the production of the consumption good,

$$g > m > 0, \tag{93}$$

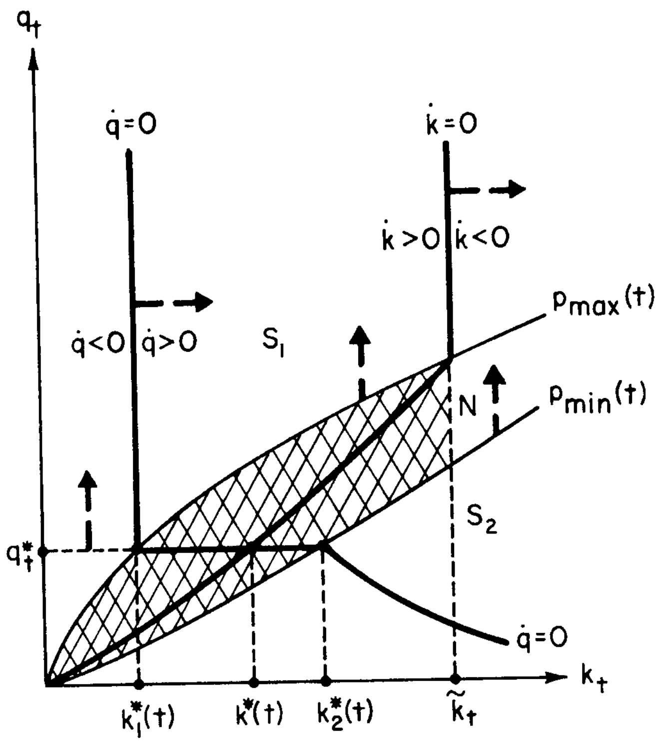

FIGURE 8. "Snapshot" at time t of the phase diagram for the case $m > g > 0$ and $k_2(\omega) > k_1(\omega)$.

is treated in similar fashion. The relevant phase diagram differs from that suggested by Figure 8 in that

$$\frac{dk_1^*(t)}{dt} > 0, \quad \text{and}$$
$$\frac{d\bar{k}(t)}{dt} > 0, \quad \text{but} \tag{94}$$
$$\frac{\partial p_{\max}(k, t)}{\partial t} < 0, \quad \text{and} \quad \frac{\partial p_{\min}(k, t)}{\partial t} < 0 \qquad \text{for } m > g > 0.$$

The special case when the rates of technical change are identical, $g = m$, is included as the final example. For this case

$$\operatorname{sgn} \frac{dk_1^*(t)}{dt} = \operatorname{sgn} \frac{d\bar{k}(t)}{dt} = \operatorname{sgn}(g),$$
$$\frac{\partial p_{\max}(k, t)}{\partial t} = \frac{\partial p_{\min}(k, t)}{\partial t} = 0, \quad \text{and thus} \tag{95}$$
$$\operatorname{sgn} \frac{dq^*(t)}{dt} = \operatorname{sgn}(g), \qquad \text{for } g = m.$$

8. The Case where Production of the Investment Good is more Capital Intensive than Production of the Consumption Good

In this section, the problem posed in section 6 is treated for the case where the production functions of Equations 31 and 32 are such that

$$k_1(\omega) > k_2(\omega) \qquad \text{for } \omega > 0. \tag{96}$$

From Equations 41 through 43 and 47, the assumption of Inequality 96 implies that $p_{max}(k, t)$ and $p_{min}(k, t)$ are strictly decreasing functions of k. Since $p_{max}(k, t) > p_{min}(k, t)$, the (k, q, t)-space is thus divided into the three mutually distinct regions S_1, S_2, and N, defined in Equations 74 through 76. As in section 7, differential Equations 77 and 78 apply for $(k, q, t) \in S_1$; Equations 79 and 80 apply for $(k, q, t) \in S_2$; Equations 81 through 83 apply for $(k, q, t) \in N$.

Again, for purposes of exposition, first consider the case of no technical change, $m = g = 0$. Combining this assumption with the assumption of Inequality 96 yields the autonomous system of differential Equations 58 and 68, whose solutions are characterized in the phase diagram of Figure 9. The loci of points yielding a stationary solution to Equations 58 and 68 are shown with heavy solid curves. The unique value of $k_1{}^*$ is determined by solving Equation 85. The unique value of q^* is found by solving

$$q^* = p_{max}(k_1{}^*).$$

By Equation 83, q^* uniquely determines ω^*, which in turn uniquely determines $k_1{}^* = k_1(\omega^*)$ and $k_2{}^* = k_2(\omega^*)$. Call the intersection of the curves determined by $\dot{q} = 0$ and $\dot{k} = 0$, k^*. Assume that the backward solutions from the point (k^*, q^*) cross the curves $p_{max}(k)$ and $p_{min}(k)$ at $k_1{}^{**}$ and $k_2{}^{**}$ respectively.

If the length of the planning period is infinite, $T = \infty$, and the "terminal" capital-labor ratio $k(\infty)$ is left free, then the optimal program of capital accumulation is easy to characterize with the aid of Figure 9.

For example, given the initial capital-labor ratio $k(0) < k_1{}^{**}$, choose $\hat{q}(0)$ such that $[k_0, \hat{q}(0)]$ lies on the backward solution from (k^*, q^*). For this case, the optimal program is specialized to production of the investment good until the critical ratio $k_1{}^{**}$ is achieved.

Consider the case where the planning period is finite and a terminal target must be met, that is,

$$0 < T < \infty \quad \text{and} \quad k(T) \geq k_T > 0.$$

The optimal program (if feasible) follows the path that traverses from k_0 to k_T in time T. For completeness, it should be remembered that if on the path so chosen $q(T) < 0$, the optimal path is instead the path that traverses from k_0 to $q(T) = 0$ in time T. If the boundary conditions k_0 and k_T are

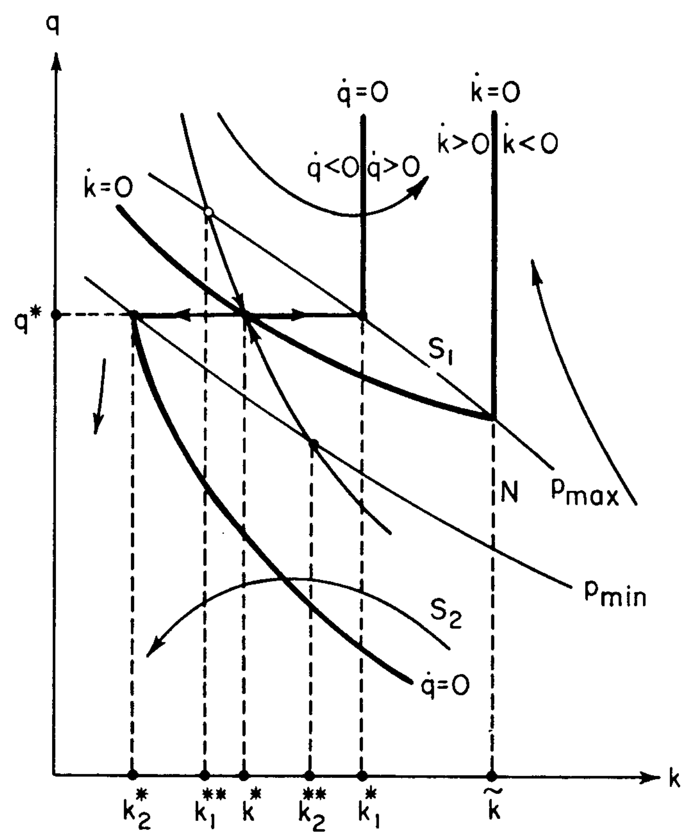

FIGURE 9. Phase diagram for the case where investment goods production is more capital intensive and where there is no technical change.

fixed as the length of the planning period is increased indefinitely, $T \to \infty$, it is seen that the optimal capital-labor profile will be arbitrarily close to the ratio k^* for all but a finite amount of time.

Next, let us turn to the case where g and m are not necessarily zero. We shall restrict our attention to the case where $T < \infty$ in order to guarantee that the value of Integral 56 is finite along all feasible paths.

As in section 7, it is convenient to suggest the appropriate three-dimensional phase diagram by a two-dimensional "snapshot" of the full diagram. The motion through time of $k_1^*(t)$ and $\tilde{k}(t)$ are given by Equations 86 and 90, respectively. Also $\partial p_{\max}(k, t)/\partial t$, $\partial p_{\min}(k, t)/\partial t$, and $(m - g)$ share the same sign.

In Figure 10, the phase diagram for the system of Equations 77 through 83 is characterized under the capital intensity assumption (Inequality 96) and under the assumption that technical progress is such that $m > g > 0$. The loci of stationary solutions to Equations 58 and 68 are shown by the heavy solid curves. Region N is indicated with crosshatching. The heavy broken arrows indicate the direction of shift of the various schedules as t increases. "Snapshots" for the various other cases may be constructed by the reader. Given the appropriate phase diagram, the optimal path is chosen by a method entirely analogous to that employed in section 7.

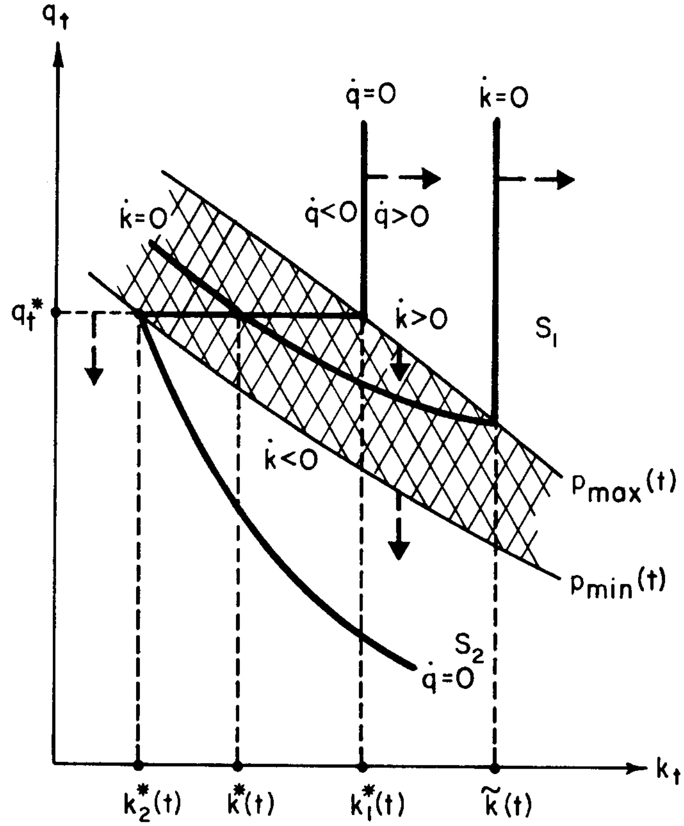

FIGURE 10. "Snapshot" at time t of the phase diagram for the case $g > m > 0$ and $k_1(\omega) > k_2(\omega)$.

9. Concluding Comments

The problem of this essay is to characterize programs of capital accumulation that maximize the discounted sum of per capita consumption over the planning period subject to the available techniques of production, given initial endowments and terminal requirements. Production is neoclassical, and Hicks-neutral technical change is autonomous and proceeds at a constant relative rate. Without too much difficulty the model can be extended to the more general case where Hicks-neutral technical change proceeds at a given (but not necessarily constant) rate.

The case where techniques are such that the production of the consumption good is always more capital intensive than production of the investment good is treated in section 7. The opposite case is treated in section 8. The degenerate case where capital intensities are always equal (the one-sector model) is treated in sections 2 through 4. The remaining case is that of reversals in factor intensities: the case where $k_2(\omega^\dagger) > k_1(\omega^\dagger)$ for some $\omega^\dagger > 0$, but where $k_1(\omega^{\dagger\dagger}) > k_2(\omega^{\dagger\dagger})$ for some $\omega^{\dagger\dagger} > 0$, $\omega^{\dagger\dagger} \neq \omega^\dagger$. The general treatment of such cases is complicated, but at least in principle the method is easily explained. At any given instant $t_1 \geq 0$, Equations 85 and 90 uniquely determine the wage-rentals ratio $\omega^*(t_1)$. The snapshot at time t_1 is constructed by the method of section 7 or 8 depending upon whether $k_2[\omega^*(t_1)] \gtrless k_1[\omega^*(t_1)]$. At another instant $t_2 \geq 0$, $\omega^*(t_2)$ may be such that the factor intensities $k_2[\omega^*(t_2)]$ and $k_1[\omega^*(t_1)]$ are reversed from the situation at time t_1. In such a case the different snapshot applies.

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