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Optimal Fiscal and Monetary Policy and Economic Growth

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1. Introduction

There have been two broad strategic approaches to the study of economic growth. The first, exemplified by Solow's paper (1956), attempts to explain how an enterprise economy will grow, given its technology and the market behavior of its consumers. The second approach, exemplified by Ramsey (1928), attempts to determine an optimal development strategy for a fully planned economy, given its technological constraints.

These approaches fail to capture a central policy problem of a modern "mixed" economy in which the government can influence investment and saving, but only indirectly, by manipulating certain basic variables like the deficit and the money supply. Our paper represents an attempt to begin the analysis of this problem.[1]

The very term "mixed economy" implies that there are two centers of decision making and that the preferences of the consumers and of the government are distinguishable.[2] It is not at all clear where the preferences of the government come from, or even whether governments have consistent preferences of the kind we will talk about. But a constant theme of policy literature is that government intervention in the economy is effective and can be judged as good or bad for the economy without direct reference to consumer preferences. This is particularly true of policy prescriptions for economic growth. It seems to us that postulating a social welfare

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[1] In a very simple dynamic model, Nelson (1966) studied the monetary and fiscal policies consistent with full employment. Using a different model, Phelps (1965*a*) has studied the effects of different monetary and fiscal policies on the level of investment and the rate of inflation.

[2] This distinction between consumer behavior and government preferences is basic to Arrow's work on social investment criteria (see Arrow, 1966).

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functional for the government is the best way to make rigorous the prescription of government control in the mixed economy.

We consider a model of a market economy with two produced commodities—consumption goods and investment goods—and three assets—money, bonds, and capital. From the usual two-sector production model we derive demands for the productive services of capital and labor, which are supplied inelastically at any moment, and supply flows of consumption and investment goods, which depend only on factor endowments and the consumption price of capital. We assume that the demand for consumption is proportional to net disposable income which includes government taxes and transfers, made in a lump-sum fashion. Given the consumption-goods price of capital and the nominal value of net government transfers, equilibrium in the market for consumption goods can be achieved only at some equilibrium consumption-goods price of money.

Given the consumption-goods price of money, the markets for stocks determine prices and rates of return for three assets: money, bonds, and capital. We do not attempt to derive demand functions for assets from individual maximizing behavior, but we believe our formulation is fairly general. Since the markets for assets and consumption goods must equilibrate simultaneously, the consumption-goods price of capital, the consumption-goods price of money, and the bond interest rate are jointly determined in these markets. The price of capital is of particular importance because it determines the flow of outputs of consumption and investment goods. Producers note the going price of capital, and they supply as much new investment as is profitable for them at that price. The new capital finds a place in portfolios through the accumulation of savings and, if necessary, through a change in the price and rate of return to all capital, old and new.

The government has already appeared twice. First, its deficit appears as transfer income and influences the demand for consumption goods. Second, the government can change the relative supplies of its own bonds and money to the asset market by making open-market purchases or sales. Changes of this kind will affect the equilibrium price of capital and will, therefore, affect the economy's growth path.

We assume the government has two goals: maximization of the integral of discounted utility of per capita consumption and the management of aggregate demand to achieve a stable consumer price level. We describe the optimal growth path for consumption and investment which, for a given welfare functional, depends only on the technology and initial capital-labor ratio, since under our assumptions these two facts are the only binding constraints on possible paths. To achieve this path while maintaining stable consumer prices, the government must manipulate its deficit and open-market policy to induce the private sector to produce investment goods at the optimal rate at each instant.

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If we compare this model with the conventional optimal growth model, we see that the private asset demands and consumption behavior are like additional constraints from the government's point of view. They are facts with which the government must operate in order to achieve its goals.

Optimality implies a specific path for the government policy variables —particularly the deficit (and thus the stock of government debt) and the composition of the debt. The mixed economy with optimal monetary and fiscal policy tends to a unique capital-labor ratio and a unique per capita government indebtedness which are independent of initial endowments.

For the special case in which the instantaneous utility function of per capita consumption has constant marginal utility, we show that the deficit increases with the capital stock along the optimal path.

We are also able to establish propositions concerning the relation between long-run optimal values for the per capita capital stock and the per capita debt for economies which have different social rates of discount and different private saving propensities. It is possible in our model for an economy with a lower social rate of discount and thus a higher long-run capital-labor ratio to have a larger long-run per capita government debt.

We draw a final conclusion which bears on the idea of the "burden of the debt." In this model the initial stock of debt has no effect at all on the optimal growth path of consumption and investment. The economy's growth possibilities are constrained only by its technology and its initial endowments of capital and labor. There is no burden to the debt per se, although the accumulation of the debt *may* have been partly at the expense of capital accumulation.

This paper represents only the beginnings of a satisfactory theory of growth policy in an indirectly controlled market economy. In particular, we expect that this analysis can be extended to models which differ somewhat in their descriptions of market behavior; for example, to models with more general consumption functions.

2. Production

Our production model is the simple two-sector constant returns-to-scale model of Uzawa (1963). The heavy curve in Figure 1 is the production possibility frontier (*PPF*) corresponding to a capital-labor ratio k. Production will take place at the point at which the *PPF* has slope $-p_k$, where p_k is the price of capital relative to consumption goods. Inspection of the figure shows that, where y_I and y_C are per capita output of investment and consumption,

$$y_I = y_I(k, p_k), \qquad \text{and} \qquad y_C = y_C(k, p_k), \tag{2.1}$$

with k and p_k uniquely determining y_I and y_C. Also,

$$\frac{\partial y_I}{\partial p_k} > 0, \qquad \text{and} \qquad \frac{\partial y_C}{\partial p_k} < 0. \tag{2.2}$$

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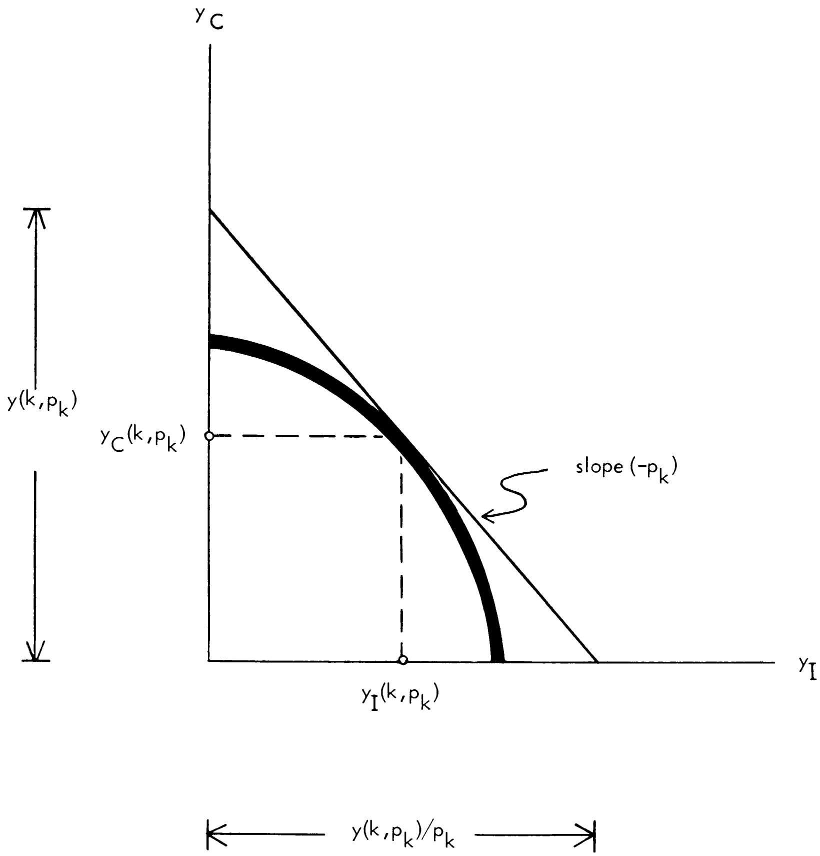

FIG. 1

Three other comparative statics propositions will be needed in our analysis. For a full discussion of these propositions, see Rybczynski (1955) or Uzawa (1963). We assume that production of consumption goods is always more capital-intensive than production of investment goods. This implies that

$$\frac{\partial y_I}{\partial k} < 0 \qquad \text{and} \qquad \frac{\partial y_C}{\partial k} > 0. \tag{2.3}$$

Another important result is that under the capital-intensity assumption, r, the rental rate on capital, depends only on p_k, the price of capital, and declines as the price of capital rises:

$$(dr/dp_k) < 0. \tag{2.4}$$

Further, the capital intensities depend only on p_k and rise as p_k rises:

$$(dk_i/dp_k) > 0, \tag{2.5}$$

where k_i is the capital intensity in sector i ($i = I, C$).

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3. Asset Market

We assume that there are three types of assets that can be held in the portfolios of wealth owners: physical capital, non-interest-bearing government debt called money, and interest-bearing government debt called bonds. A unit of capital yields a return ρ_k given by

$$\rho_k = r(p_k)/p_k + \pi_k, \tag{3.1}$$

where r is the rental rate on capital and π_k is the rate at which individuals expect p_k to change. In section 2 we noted that r is a decreasing function of p_k.

We assume money yields no interest payment, so ρ_m, its rate of return, is given by

$$\rho_m = \pi_m, \tag{3.2}$$

where π_m is the expected rate of change in the consumption-goods price of money, ρ_m.

For simplicity, we assume that bonds (like savings deposits) have variable income streams but that their money price remains constant. By a proper choice of units, we can set the price of bonds, p_b equal to p_m. Thus the rate of return on bonds is

$$\rho_b = i + \pi_m, \tag{3.3}$$

where i is the bond rate of interest.

At each moment of time, the real per capita quantities of capital (kp_k), money (mp_m), and net holdings of government bonds (bp_m) that wealth owners desire to hold in their portfolios depend upon their real per capita wealth (a); upon the rates of return on capital (ρ_k), on money (ρ_m), and on bonds (ρ_b); and upon the consumption value of per capita output ($y = y_C + p_k y_I$, representing the transactions motive for holding assets).

When the asset market is in equilibrium:

$$kp_k = J(a, y, \rho_k, \rho_m, \rho_b), \tag{3.4}$$

$$m\rho_m = L(a, y, \rho_k, \rho_m, \rho_b), \tag{3.5}$$

$$bp_m = H(a, y, \rho_k, \rho_m, \rho_b), \tag{3.6}$$

with

$$a = kp_k + (b + m)p_m = kp_k + gp_m, \tag{3.7}$$

where g is the aggregate per capita stock of government debt. By Walras' Law, applied to the individual wealth constraint (3.7), if any two of the three asset markets are in equilibrium, the third one must also be in equilibrium. Therefore, given π_k, p_m, and π_m, and the supplies of assets, any two of equations (3.4)–(3.6) together with (3.7) determine the equilibrium price of capital p_k and the bond rate of interest i. We assume that

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assets are gross substitutes for each other and that none of the assets is inferior, that is, that wealth elasticities of demand are positive.

We assume that only the government can issue money, so the net per capita holdings of money in private portfolios must be non-negative, or, $m \geq 0$. On the other hand, the private sector can issue interest-bearing debt, and the government could choose to be a net holder of bonds, allowing b to be negative. In fact, net government bond holdings could be sufficiently large to cause the government to be a net creditor[3] when $g = b + m < 0$.

Through open-market purchases and sales, the government determines the composition of its outstanding debt. We call the debt-money ratio $x = g/m$. An open-market purchase increases the supply of money and leaves g unchanged, thereby lowering x when $g > 0$.

Proposition 1

An open-market purchase increases the equilibrium price of capital p_k and lowers the equilibrium rate of interest i.

Proof

By substituting $x = g/m = (b + m)/m$ in (3.4) and (3.5), and implicitly differentiating we obtain

$$\frac{\partial p_k}{\partial x} = \frac{-g p_m(\partial J/\partial \rho_b)}{x^2 \Delta} \lesseqgtr 0 \text{ as } g \gtrless 0, \tag{3.8}$$

where $\Delta = (\partial L/\partial p_k)(\partial J/\partial \rho_b) - [(\partial J)/(\partial p_k) - k](\partial L/\partial \rho_b) < 0$, since by assumption $(\partial L/\partial p_k) = k(\partial L/\partial a) + (\partial L/\partial y)(\partial y/\partial p_k) + (\partial L/\partial \rho_k)(\partial \rho_k/\partial p_k) > 0$, and $(\partial J/\partial p_k) - k = k[(\partial J/\partial a) - 1] + (\partial J/\partial y)(\partial y/\partial p_k) + (\partial J/\partial \rho_k)(\partial \rho_k/\partial p_k) < 0$. Similarly $\partial i/\partial x \gtrless 0$ as $g \gtrless 0$.

An open-market purchase forces a change in equilibrium asset prices and rates of return. In particular, when π_k, p_m, π_m, and k are held constant, the rate of interest i will have to fall in order to induce wealth owners to hold a larger amount of money and a smaller amount of bonds in their portfolios. The fall in the rate of interest increases the demand for capital, thus leading to an increase in the price of capital. This heuristic argument is formalized in the comparative statics of Proposition 1.

Next, we make an important assumption. We assume that the demand functions $J(\cdot)$, $L(\cdot)$, and $H(\cdot)$ are sufficiently "flexible" so that, given g and $k > 0$, the government, by setting the current level of x, will be able to set the price p_k at any level consistent with tangency of the national income

[3] Remember that certain of our government agencies, such as the FNMA, are net creditors to the public.

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isoquant with the *PPF* in Figure 1. Thus, we assume that by varying the debt-money ratio the government is able to achieve any efficient mix of consumption and investment.[4] The most likely difficulty in meeting this requirement is that there might be some rate of return to capital so low that increasing m will lead to no rise in p_k. What we are ruling out here is a kind of "liquidity trap," since we assume monetary policy is at least potent enough to achieve any value of p_k for which production is not completely specialized.

4. Saving and Growth

The government issues debt in order to finance its budget deficit. Let d denote the per capita government deficit; then

$$\dot{g} = d - ng, \tag{4.1}$$

where n is the relative rate of population growth.

We define per capita net disposable income, $\hat{y}$, as the value of factor payments (equal to the value of output), net government transfers to the private sector, and expected asset appreciation.[5] Since we assume that there are no government expenditures, the value of net government transfers is equal to the government budget deficit, so that

$$\hat{y} = y_C + p_k y_I + p_m d + p_m \pi_m g + p_k \pi_k k. \tag{4.2}$$

In what follows, we simplify by assuming that $\pi_k = 0$.[6] We also assume that $\pi_m = 0$ because we will study only situations in which the government manipulates fiscal and monetary policy to achieve a constant price level and therefore a constant p_m.

We further assume that individuals save a constant fraction s of income, $\hat{y}$, so that, if $\pi_k = 0 = \pi_m$, for the commodity market to be in equilibrium

$$y_C = (1 - s)\hat{y} = (1 - s)(y + p_m d), \tag{4.3}$$

where $y = y_C + p_k y_I$.

[4] Alternatively, the assumption is equivalent to saying that by merely varying x, the government will be able to trace out all points on the *PPF* of Figure 1. The comparative statics and comparative dynamics of the descriptive model are treated in greater detail in Foley and Sidrauski (1967).

[5] See Shell, Sidrauski, and Stiglitz (1969), where $\hat{y}$ is called per capita Individual Purchasing Power.

[6] While in the short run p_k may change, in our analysis $\dot{p}_k$ tends to zero in the long run, thus lending some justification to our assumption. Also, since we will be concentrating on the partially controlled economy in which the government possesses long-run foresight, problems of instability à la Hahn (1966) and Shell and Stiglitz (1967) will not arise—no matter how individuals form expectations about price changes.

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At any moment, k and g are historically given by inherited endowments. If the government sets the debt-money ratio at x, then we can think of the price of capital p_k as being determined in the asset market, that is, by equations (3.4)–(3.7). Given k and p_k, producers determine y_C, y_I, and thus y.

Then there are two ways in which we can view equation (4.3). If the per capita deficit is d, equation (4.3) can be solved for the price of money, p_m, that will clear the commodity market. Alternatively, if the government wants to sustain some price level $(1/p_m^o)$, then (4.3) can be solved for that per capita deficit d that will clear the commodity market when p_m is held equal to p_m^o. In what follows, we consider the case in which the government is committed to pursuing a constant price level policy—a policy which sustains forever the initial price level $(1/p_m^o)$, so $\dot{p}_m = 0$. Then from (4.3)

$$d = \phi(k, p_k). \tag{4.4}$$

Substituting (4.4) in (4.3) and differentiating yields,

$$\frac{\partial \phi}{\partial k} = \frac{1}{p_m^o}\left(\frac{s}{1-s}\frac{\partial y_C}{\partial k} - p_k \frac{\partial y_I}{\partial k}\right) > 0, \tag{4.5}$$

by (2.3) and

$$\frac{\partial \phi}{\partial p_k} = \frac{1}{p_m^o}\left(\frac{s}{1-s}\frac{\partial y_C}{\partial p_k} - p_k \frac{\partial y_I}{\partial p_k} - y_I\right) < 0, \tag{4.6}$$

by (2.2). Under our capital-intensity hypothesis, when k rises, per capita output of consumption goods y_C rises faster than per capita national product y. Therefore, as k rises, the government must increase the per capita deficit ϕ in order to close this deflationary gap. That is, the higher the capital-labor ratio k, the higher is the per capita deficit ϕ which is required to stabilize the consumer price level. No matter what capital-intensity assumption is made, the higher the price of capital, the higher is national product, while the lower is the output of consumption goods. Therefore, as p_k rises, the government must decrease its deficit in order to close this inflationary gap. That is, the higher the price of capital p_k, the lower the per capita deficit ϕ which is required to stabilize the consumer price level.

Since p_k and k uniquely determine y_I, capital accumulation is given by

$$\dot{k} = y_I(k, p_k) - nk. \tag{4.7}$$

Equation (4.7) can be rewritten as

$$\dot{k} = \frac{sy - (1-s)dp_m^o}{p_k} - nk. \tag{4.8}$$

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5. Optimal Growth in the Fully Controlled Economy[7]

Suppose that the central planner can directly command the allocation of resources. Suppose also that the planner's notion of instantaneous welfare is based exclusively on per capita consumption y_C. Notably, the planner is assumed to take no direct account of the population's asset preferences. We can assume that the planner seeks to maximize the intertemporal welfare functional,

$$\int_0^{\infty} U[y_C(t)]e^{-\delta t}dt, \tag{5.1}$$

where marginal utility, U', is positive but declining; $U'' < 0$. The planner's pure subjective rate of time discount is $\delta > 0$. Choosing utility as the numeraire, socially valued, discounted, per capita national product H is given by

$$H = [U(y_C) + q(y_I - nk)]e^{-\delta t}, \tag{5.2}$$

where q is the demand price of investment goods in terms of utility currently foregone. For a program to be feasible

$$\dot{k} = y_I - nk \qquad \text{and} \qquad k(0) = k_0, \tag{5.3}$$

where k_0 is the inherited capital-labor ratio. For a program to maximize (5.1) subject to the two-sector technology and (5.3), y_C and y_I be chosen at each instant so as to maximize national product H. Thus $U'(y_C) + q(dy_I/dy_C)_{PPF} = 0$, which is equivalent to

$$U'(y_C) = q/p_k. \tag{5.4}$$

The first-order condition (5.4) states that the marginal utility of per capita consumption must be set equal to the utility price of investment divided by the consumption price of investment.[8]

Along the optimal trajectory, the social return on a unit of capital must be equal to the discount rate δ,

$$(\dot{q}/q) + r[k_I(p_k)] = \delta + n, \tag{5.5}$$

where k_I is the *efficient* capital-labor ratio in the investment goods industry when the consumption price of capital is p_k.[9]

[7] This section is essentially a review of established results in the two-sector, optimal-growth theory (see, esp., Cass [1965]). Analysis of the case with a linear objective functional appears in Uzawa (1964) and Shell (1967).

[8] In (5.4) it is assumed that on an optimal trajectory the *utility* demand price of investment is equal to the *utility* supply price of investment. This assumption means that the optimal allocation is not completely specialized. For the fuller analysis, see Uzawa (1964), Cass (1965, esp. chap. iii), and Shell (1967).

[9] Compare with Pontryagin, *et al.* (1962). The condition is that $d(qe^{-\delta t})/dt = -\partial H/\partial k$, which reduces to (5.5) after applying an envelope theorem to the expression (dy/dk). The population growth rate n appears on the *RHS* of (5.5) because q is the

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It is further required for optimality that the present discounted utility value of the per capita capital stock tend to zero, so that

$$\lim_{t \to \infty} q(t)e^{-\delta t}k(t) = 0. \tag{5.6}$$

From (5.5), $\dot{q} = 0$ when

$$r[k_I(p_k)] = \delta + n. \tag{5.7}$$

Since we assume that efficient capital intensities do not cross, we know from the two-sector production model that (5.7) holds for a unique consumption price of investment p_k^*. So that from (5.4), $\dot{q} = 0$ if $q = p_k^* U'(y_C)$, yielding

$$\left(\frac{dq}{dk}\right)_{\dot{q}=0} = p_k^* U''(y_C) \frac{\partial y_C}{\partial k} < 0 \tag{5.8}$$

by (2.3).

From (5.3), $\dot{k} = 0$ when $k = y_I/n$, it can be shown that the $\dot{k} = 0$ curve crosses the $\dot{q} = 0$ curve from the left.[10] Their unique intersection is labeled (k^*, q^*). It also follows that

$$\lim_{k \to 0} q \Big|_{\dot{k}=0} = \infty \qquad \text{and} \qquad \lim_{k \to \tilde{k}} q \Big|_{\dot{k}=0} = \infty,$$

where $\tilde{k}$ is the maximum sustainable capital-labor ratio.

The laws of motion for the system (5.3) and (5.5) are described in the phase diagram of Figure 2. The unique stationary point (k^*, q^*) is a saddlepoint. The heavy arrows indicate the locus of points (k, q) tending to (k^*, q^*). Trajectories not tending to (k^*, q^*) can be shown to violate (5.6). Therefore, given the initial capital-labor ratio k_0, $q(0)$ is uniquely chosen by the planner, so that $[k_0, q(0)]$ lies on the heavy curve in Figure 2.

Several properties of the optimal solution are noteworthy.

Proposition 2

On an optimal path, the long-run marginal product of capital is equal to the rate of population growth plus the rate of time discount.

This follows immediately from considering the stationary solution to (5.5). Thus, as the rate of discount becomes small, $\delta \to 0$, the optimal long-run marginal product of capital approaches the rate of growth, and the long-run capital-labor ratio approaches the Golden Rule value.

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utility price of a unit of k rather than K. The social rate of return to K must equal δ; thus the rate of return to k must equal $\delta + n$. Equation (5.5) has some implications for decentralization. If the price system $q(t)$ obtains, factors are rewarded by marginal products, and the government sells, for a unit of utility, a consol that pays the instantaneous rate δ. (5.5) is then the perfect-foresight, asset-market clearing equation.

[10] See Cass (1965).

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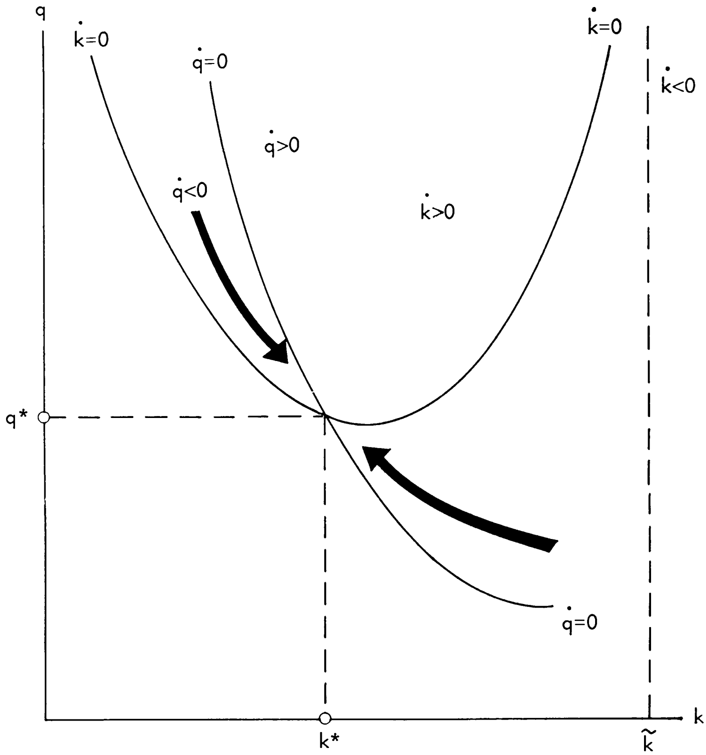

FIG. 2

Proposition 3

If the initial capital-labor ratio k_0 is less than the long-run optimal capital-labor ratio k^*, then along the optimal trajectory: (1) k is increasing through time, and (2) the utility demand price of investment q is decreasing through time. If the initial capital-labor ratio k_0 is greater than the long-run optimal capital-labor ratio k^*, then along the optimal trajectory: (1) k is decreasing, and (2) q is increasing.

Proposition 3 implies that along an optimal trajectory (opt.), sign($\dot{k}$) = sign($-\dot{q}$) = sign($k^* - k$). That is,

$$(dq/dk)_{\text{opt.}} < 0. \tag{5.9}$$

Proposition 4

(1) If the initial capital-labor ratio k_0 is less than k^*, then on the optimal trajectory per capita consumption is increasing. (2) If the initial capital-labor ratio is greater than k^*, then on the optimal trajectory per capita consumption is decreasing.

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Proof

Time differentiation of y_C yields

$$\dot{y}_C = \frac{\partial y_C}{\partial k}\dot{k} + \frac{\partial y_C}{\partial p_k}\left(\frac{\partial p_k}{\partial q}\dot{q} + \frac{\partial p_k}{\partial k}\dot{k}\right), \tag{5.10}$$

$$\frac{\partial p_k}{\partial q} = \frac{1}{\left[\frac{q}{p_k} + U''(y_C)\frac{\partial y_C}{\partial p_k}p_k\right]} > 0 \tag{5.11}$$

by (2.2), and

$$\frac{\partial p_k}{\partial k} = \frac{-U''(y_C)\frac{\partial y_C}{\partial k}}{\left[U''(y_C)\frac{\partial y_C}{\partial p_k} + \frac{q}{p_k^2}\right]} > 0 \tag{5.12}$$

by (2.2) and (2.3). Combining (5.10)–(5.12) yields

$$\dot{y}_C = \frac{\partial y_C}{\partial k}\left[\frac{q}{p_k^2 U''(y_C)\frac{\partial y_C}{\partial p_k} + \frac{q}{p_k^2}}\right]\dot{k} + \frac{\partial y_C}{\partial p_k}\frac{\partial p_k}{\partial q}\dot{q} \gtreqless 0 \text{ as } k \lesseqgtr k^* \tag{5.13}$$

by Proposition 3.

Proposition 5

On an optimal trajectory, (1) if $k < k^*$, then $p_k < p_k^*$, while (2) if $k > k^*$, then $p_k > p_k^*$.

Proof

If and only if $\dot{q} = 0$, $p_k = p_k^*$. But for $k < k^*$, the optimal trajectory lies below the $\dot{q} = 0$ curve. Therefore, by (5.11), $p_k < p_k^*$. Similarly for $k > k^*$, $p_k > p_k^*$.

Since at k^* the optimal consumption price of investment is p_k^*, an immediate corollary to Proposition 5 is that on an optimal path,

$$\left(\frac{dp_k}{dk}\right)_{\substack{\text{opt.}\\ k=k^*}} > 0. \tag{5.14}$$

Notice, however, that it is not necessarily true that p_k is monotonic in k on an optimal path.

In the analysis of this section, we have so far used an important curvature assumption: $U''(y_C) < 0$, which implies that the instantaneous preference map in (y_I, y_C) space is *strictly* convex.[11] In order to study the limiting behavior of our model, we will relax this assumption in what follows.

[11] The equation "$U(y_C) + qy_I =$ a constant" describes the relevant indifference curve.

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Proposition 6

If, in the preceding analysis (eqs. [5.1]–[5.14]), we replace the assumption $U''(y_C) < 0$ with the assumption that $U''(y_C) = 0$, then *along the non-specialized segment of the optimal trajectory* the price of capital p_k is a constant, independent of the capital-labor ratio k, and $(dp_k/dk)_{\text{opt.}} = 0$.

Proof

Maximization of national income implies from (5.4) that

$$q = \beta p_k, \tag{5.15}$$

where the constant $\beta \equiv U' > 0$. But from (5.5), $\dot{q} = 0$ only if k_I^* is the unique root to (5.7). From (2.5), k_I^* uniquely determines the consumption price of capital p_k^*, which in turn uniquely determines the utility price of capital q^*, from (5.15). If $p_k < p_k^*$, then by (2.8) and (5.5) q must fall at a rate faster than $\delta + n$. If $p_k > p_k^*$, then q must rise at a rate faster than $\delta + n$. Therefore, in order for the condition (5.6) to hold, on an optimal (non-specialized) trajectory, p_k must be constant and equal to (β/q^*).[12]

6. Optimal Fiscal and Monetary Policy

We return to the analysis of the mixed economy described in sections 2–4. The government is assumed to have two goals: (1) the maintenance of price stability, and (2) the constrained maximization of a utility functional based on the stream of per capita consumption. At each moment, the government possesses two policy tools that it can employ in pursuit of these goals: (1) the composition of the government debt (*monetary policy*) which is reflected in the debt-money ratio x, and (2) the size of the per capita government deficit d (*fiscal policy*).[13] Government action is constrained by the behavior of producers (described in section 2), by the behavior of asset holders (described in section 3), and by the saving behavior of individuals (described in section 4).

Formally, the government chooses time paths for $x(t)$ and $d(t)$ that maximize the welfare functional

$$\int_0^\infty U[y_C(t)]e^{-\delta t}dt,$$

subject to the policy constraint that $p_m(t) = p_m^o$ for all $t \geq 0$. From section

[12] A complete discussion of this model with a linear criterion functional appears in Uzawa (1964) and Shell (1967), where, by convention, $\beta \equiv 1$. A fuller account of the nature of the "transversality condition" (5.6) appears in Shell (1969).

[13] In the terminology of Tinbergen, the government has two targets, and it has two instruments it can employ in the pursuit of those targets.

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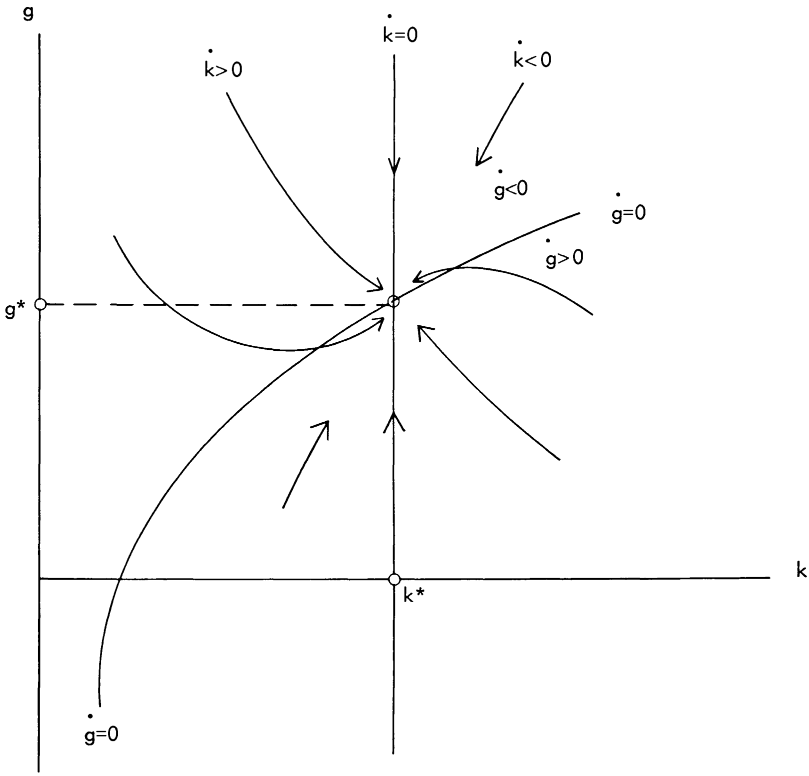

FIG. 3

5, we know that on an optimal trajectory the price of capital p_k is a function only of the capital-labor ratio k. We can thus write equation (4.7) as

$$\dot{k} = y_I[k, p_k(k)] - nk. \tag{6.1}$$

Since the government maintains a stable price level, from (4.1) and (4.4) the change in the per capita government debt on an optimal path is given by

$$\dot{g} = \phi[k, p_k(k)] - ng. \tag{6.2}$$

The dynamic behavior of the mixed economy with optimal monetary and fiscal policy is completely described by the system (6.1)–(6.2). The capital-labor ratio k uniquely determines the optimal consumption price of capital $p_k(k)$ (as described in section 5). At the given target p_m^o, the government must choose x instantaneously so that the asset market equilibrium p_k is the same as the optimal $p_k(k)$. Given k and p_k together, producers determine the supplies of investment and consumption goods. The government then must adjust its deficit so that the market for consumption goods also clears at the given target p_m^o.

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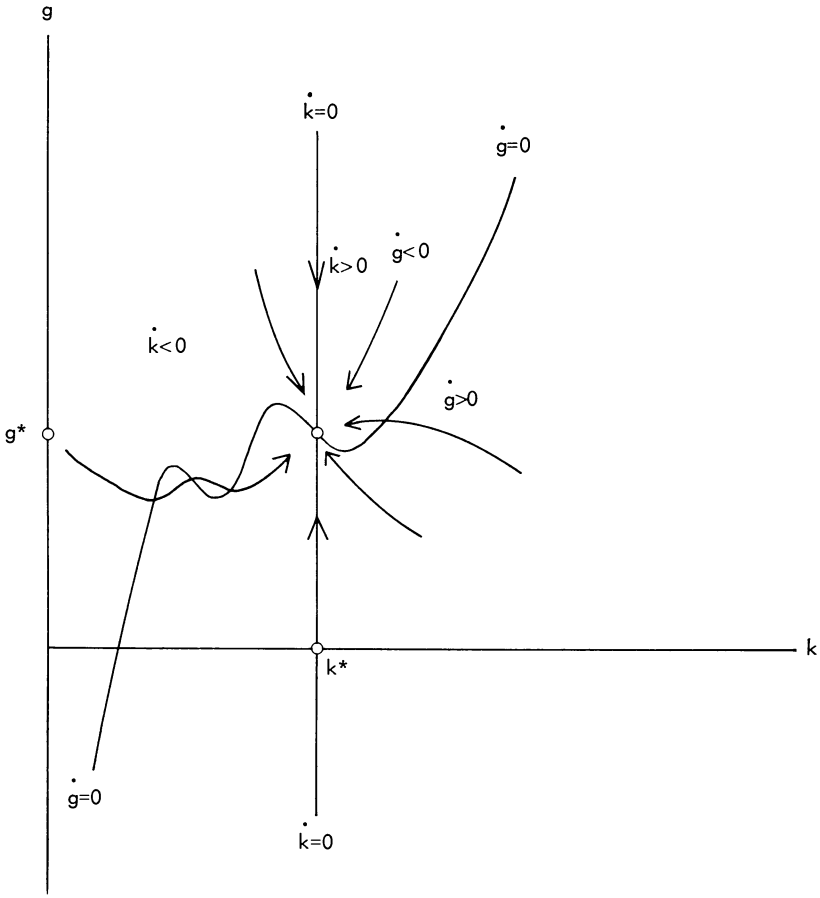

FIG. 4

From Proposition 2, we know that the long-run optimal capital-labor ratio k^* is unique. We also know by Proposition 3 that for $k < k^*$, $\dot{k} > 0$; for $k > k^*$, $\dot{k} < 0$. Thus in the phase diagrams of Figures 3 and 4, $\dot{k} = 0$ only on a vertical line in the (g, k) plane.

From (6.2), we know that $\dot{g} = 0$ if, and only if,

$$g = \phi[k, p_k(k)]/n. \tag{6.3}$$

In order to find the slope of the curve described in (6.3), we totally differentiate (4.4) *along an optimal trajectory*, yielding

$$p_m^o \frac{d\phi}{dk} = \frac{s}{1-s}\frac{\partial y_C}{\partial k} + \frac{s}{1-s}\frac{\partial y_C}{\partial p_k}\left(\frac{dp_k}{dk}\right)_{\text{opt.}} - p_k \frac{\partial y_I}{\partial k} - p_k \frac{\partial y_I}{\partial p_k}\left(\frac{dp_k}{dk}\right)_{\text{opt.}} - y_I\left(\frac{dp_k}{dk}\right)_{\text{opt.}} \tag{6.4}$$

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from the commodity-market clearing equation (4.3). Combining (2.2) and (2.3) with Proposition 5 yields the result that $(d\phi/dk)$ may be either positive or negative in the case with $U''(y_C) < 0$. However, in the limiting case of the linear criterion functional where $U''(y_C) = 0$, we can show that the optimal per capita deficit $\phi(\cdot)$ is an increasing function of the capital-labor ratio. This is a consequence of the fact that in this case p_k is constant on the optimal path, so as k rises, the per capita output of consumption goods rises faster than per capita national product. The growing deficit is necessary to close this deflationary gap.

Proposition 7

In the case where the objective functional is linear in per capita consumption, $U''(y_C) = 0$, $(d\phi/dk) > 0$.

Proof

Again, restricting our attention to cases of non-specialization, Proposition 6 yields $(dp_k/dk)_{\text{opt.}} = 0$. Since

$$(1 - s)\phi p_m^o = sy_C - (1 - s)p_k y_I,$$

$$p_m^o\left(\frac{d\phi}{dk}\right)_{\text{opt.}} = \frac{s}{1 - s}\frac{\partial y_C}{\partial k} - p_k\frac{\partial y_I}{\partial k} > 0$$

by equation (2.3).

Also, since the right-hand sides of (6.1) and (6.2) are continuously differentiable, when the objective functional is "nearly" linear the optimal per capita deficit is an increasing function of the capital-labor ratio.

On the assumption that $(d\phi/dk) > 0$, which implies that $(dg/dk)_{\dot{g}=0} > 0$, the system in equations (6.1) and (6.2) is described in the phase diagram of Figure 3. The stationary (balanced growth) solution to (6.1) and (6.2), that is (g^*, k^*), is unique. There is no reason, however, that g^* must be positive. The question is whether the private sector will save too much or too little at the long-run optimal per capita national product y^* to maintain the long-run optimal capital-labor ratio k^*. If it saves too little,[14] $(sy^*/p_k^*) < nk^*$, the government will be forced to make up for this by running a surplus $(d^* < 0)$, and in the long run a surplus implies some indebtedness of the citizens to the government $(g^* < 0)$. Both the surplus and the net indebtedness will grow in absolute value at the same rate as the population. If the community saves too much, $(sy^*/p_k^*) > nk^*$, the government is forced to dissave constantly through a deficit $(d^* > 0)$ that maintains a constant per capita stock of debt $(g^* > 0)$.

[14] See equation (4.8).

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We do not need to restrict our attention to the case where $(d\phi/dk) > 0$. In Figure 4, we describe the behavior of the mixed economy for the case where $(d\phi/dk)$ changes sign. In both cases (Figs. 3 and 4), the balanced growth equilibrium (g^*, k^*) is globally stable.

We now describe in detail the development of the mixed economy with optimal fiscal and monetary policy, which is described by equations (6.1) and (6.2).

Proposition 8

(1) The balanced growth state (g^*, k^*) is unique and is globally stable. That is, the mixed economy with optimal fiscal and monetary policy tends to (g^*, k^*) independently of initial endowments (g_0, k_0). (2) On an optimal trajectory, k is either monotonically decreasing or monotonically increasing. (3) If the criterion functional is linear, $U''(y_C) = 0$, or nearly linear, that is,

$$\min_{y_C} U''(y_C)$$

is close enough to zero so that $(d\phi/dk) > 0$, then $(dg/dt)_{\text{opt.}}$ changes sign at most once. (4) In any case, however, $x(t)$ tends to some long-run limit x^*, and there exists a time T after which $g(t)$ is monotonic.

Proof

(1) Uniqueness follows from the fact that ϕ is a single-valued function of k. From Proposition 3, $\dot{k} = \psi(k)$, where, in the neighborhood of k^*, $\psi(\cdot)$ is a decreasing function. Taking a linear approximation to (6.1) and (6.2), about (g^*, k^*) yields the associated characteristic equation

$$x^2 + [n - \psi'(k^*)]x - n\psi'(k^*) = 0,$$

where x is the characteristic root. Since the sum of the roots is negative while the product of the roots is positive, (g^*, k^*) is locally stable and thus globally stable; (2) follows directly from Proposition 3; (3) follows from equation (6.4), Proposition 7, the continuous differentiability of (6.1) and (6.2), and from Figure 3. (4) Since $p_k(t)$, $g(t)$, and $k(t)$ tend to limits, $x(t)$ also tends to a proper limit. We assume that the production functions are well behaved, and therefore the $\dot{g} = 0$ curve is well behaved—having a finite number of local extrema in any finite interval. Hence, there must exist $\epsilon > 0$ such that $\dot{g} = 0$ is monotonic in the region $[k^* - \epsilon, k^*]$ and in the region $[k^*, k^* + \epsilon]$. Since an optimal trajectory spends all but a finite amount of time in one of these two regions, there must exist $0 < T < \infty$, such that for $t > T$, $g(t)$ is monotonic.

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7. Burden of the Debt and Comparative Dynamics

From Proposition 8, we conclude that *in our model, there is no burden of the government debt per se*. Not only is the long-run debt g^* independent of initial debt g_o, but the entire trajectory of per capita consumption (and thus welfare) is also independent of the level of the debt that the economy inherits. The real consumption opportunity that is left to a generation is entirely described by its inherited capital-labor ratio and is unaffected by its inherited government indebtedness.[15]

The conclusion that there is no burden of the government debt per se does not depend upon whether or not the saving of the private sector depends upon the private sector's wealth. This conclusion, however, is crucially dependent upon the assumption made in section 3 that asset demand functions $H(\cdot)$, $J(\cdot)$, and $L(\cdot)$ are sufficiently flexible so that in holding the price level constant, the government, by monetary and fiscal policy, is able to achieve any efficient allocation of output regardless of the value of g. If such flexibility does not exist, then the analysis of optimal fiscal and monetary policy outlined in section 6 would need to be altered. Without this flexibility in the asset market, the government would have to pursue a "second-best" fiscal and monetary policy—achieving less welfare than is possible in the fully controlled economy. In this case, the government would also have to consider the trade off between relative stability of the consumer price level and the current and future consumption opportunities of the economy.

We now develop certain propositions in comparative dynamics for the mixed economy in which the government pursues the "first-best" optimal fiscal and monetary policy described in section 6.

Proposition 9

Given technology, the long-run optimal capital-labor ratio k^* depends solely upon the government's pure rate of time discount δ. The more impatient the government (that is, the higher δ), the lower the long-run optimal capital-labor ratio k^*; $(dk^*/d\delta) < 0$.

Proof

The proof is based on Figure 2. In the (q, k) plane, the $\dot{k} = 0$ schedule is independent of δ. However, from (5.5) and (5.7), we have along the $\dot{q} = 0$ schedule, the higher δ, the lower is k_I. From (2.5), it follows that on the $\dot{q} = 0$ schedule, $(\partial p_k/\partial \delta) < 0$. From (5.11) and (5.12), it follows that as δ increases, the $\dot{q} = 0$ must shift to the southwest. The result follows immediately.

[15] The accumulation of that debt *may* have been at the expense of capital accumulation prior to the beginning of the planning period (see Proposition 10).

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Proposition 9 also applies to the limiting case of the linear criterion functional. In this case, non-specialized maximization of national income H requires that $q = \beta p_k$, where the constant $\beta = U' > 0$. Since (5.5) and (5.7) tell us that $\dot{q} = 0$ for the unique k_I^* that solves $f_I'(k_I) = \delta + n$, we have that $\dot{q} = 0$ for a unique consumption price of capital p_k^* and utility price of capital q^*. Therefore, $\dot{q} = 0$ only on a horizontal line in the (q, k) plane of Figure 2. Since in this case, $U'' = 0$, (5.12) yields $(\partial p_k/\partial k) = 0$ so the $\dot{k} = 0$ curve is independent of δ and has a positive slope. As δ increases, the $\dot{q} = 0$ line shifts to the south, while the $\dot{k} = 0$ curve does not shift. Thus, $(\partial k^*/\partial \delta) < 0$.

Proposition 10

The derivative $(\partial g^*/\partial \delta)$ is either positive, negative, or zero depending upon the technology and the private sector's savings propensity s, and the government's rate of time discount δ.

Proof

Setting $\dot{k} = 0$ and differentiating in (4.7) yields

$$\frac{\partial p_k^*}{\partial k^*} = \frac{n - (\partial y_I/\partial k)}{(\partial y_I/\partial p_k)} > 0.$$

Since

$$\frac{d\phi^*}{dk^*} = \frac{\partial \phi}{\partial k} + \frac{\partial \phi}{\partial p_k}\frac{\partial p_k^*}{\partial k^*},$$

we have, from (4.5) and (4.6), that

$$\frac{d\phi^*}{dk^*} = \frac{1}{p_m^o}\left[\left(\frac{s}{1-s}\right)\left(\frac{\partial y_C}{\partial k}\right) - p_k\frac{\partial y_I}{\partial k} + \left(\frac{s}{1-s}\right)\frac{\partial y_C}{\partial p_k}\frac{dp_k^*}{dk^*} - p_k\frac{\partial y_I}{\partial p_k}\frac{dp_k^*}{dk^*} - y_I\frac{dp_k^*}{dk^*}\right],$$

which may be either positive, negative, or zero. That is, from (4.5) and (4.6), $(d\phi^*/dk^*) \gtreqless 0$ as $(\partial\phi/\partial k) \gtreqless -(\partial\phi/\partial p_k)(\partial p_k^*/\partial k^*)$. The less impatient society (that is, the smaller δ), the greater is the long-run optimal capital-labor ratio k^* and, therefore, the greater is the price of capital p_k^* necessary to sustain that capital-labor ratio. *Ceteris paribus*, the greater the capital-labor ratio k^*, the greater is the per capita deficit ϕ^* which maintains a stable consumer price level (equation [4.5]). But, *ceteris paribus*, the greater the price of capital p_k^*, the smaller the per capita deficit ϕ^* which maintains a stable consumer price level (equation [4.6]). We can actually exhibit separate cases in which each of these effects dominates.

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From (4.3), we have $p_m^o(d\phi^*/dk^*) = [(dy_C^*/dk^*)/(1 - s)] - (dy^*/dk^*)$. The higher k^*, the higher are steady-state per capita consumption y_C^*,[16] steady-state per capita investment y_I^*, steady-state price of capital p_k^*, and steady-state per capita output $y^* = y_C^* + p_k^* y_I^*$. Thus, holding all other parameters constant, for s sufficiently close to unity $(d\phi^*/dk^*) > 0$. Since $(dy^*/dk^*) = (dy_C^*/dk^*) + p_k^*(dy_I^*/dk^*) + y_I^*(dp_k^*/dk^*) > (dy_C^*/dk^*)$, there must exist a positive s sufficiently close to zero so that $(d\phi^*/dk^*) < 0$.

Notice that this proposition is independent of the factor-intensity assumption (that is, whether k_C is greater than, equal to, or less than k_I). Indeed, for the one-sector model (the equal-factor-intensity case where $p_k^* \equiv 1$), we have $p_m^o(d\phi^*/dk^*) = (s[r^* - n])/(1 - s) - n = s\delta/(1 - s) - n$. Again for s sufficiently close to unity, $(d\phi^*/dk^*) > 0$. For s sufficiently close to zero, $(d\phi^*/dk^*) < 0$.

Proposition 10 tells us that if we compare two economies with the same technology and the same individual savings behavior (equal s), the economy whose government is less impatient (the government with the smaller δ) may seek a higher long-run per capita government debt g^*. Thus, depending upon technology and individual savings behavior, *the economy seeking the higher long-run per capita consumption* y_C^* *(and thus the higher long-run capital-labor ratio* k^**) may follow a fiscal and monetary policy leading to a higher long-run per-capita debt* g^*. Proposition 10 contrasts sharply with the widely held belief that the larger the long-run debt, the lower is the long-run capital stock.

Proposition 11

Taking the government's objective functional as given, the higher the community's savings propensity s, the higher is the long-run debt g^*, $(\partial g^*/\partial s) > 0$.

Proof

From (4.3), $(1 - s)\phi^* p_m^o = y_C^* - (1 - s)y^*$, where asterisks are used to indicate long-run equilibrium values of variables. Differentiating yields $p_m^o(\partial\phi^*/\partial s) = y_C^*/(1 - s)^2 > 0$. The proposition follows immediately from (4.1).

This leads to the natural classification of economies into those which are long-run "oversavers" and those which are long-run "undersavers." Given the government's discount rate δ, if s is sufficiently large, then long-run fiscal policy will lead the government to a net debtor position ($g^* > 0$). If s is sufficiently small, then the government will become a net creditor ($g^* < 0$).

[16] Since optimal programs are dynamically efficient, k^* is not greater than the Golden Rule capital-labor ratio, $(r^*/p_k^*) \geq n$. Therefore, $(dy_C^*/dk^*) > 0$.

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By continuity for every $\delta > 0$, there must exist some s ($0 < s < 1$) such that $g^* = 0$.

The converse is not true. If s is sufficiently high, then a government policy in which

$$\lim_{t \to \infty} g(t) = 0$$

will lead to a capital-labor ratio forever bounded above the Golden Rule. We know by the Phelps-Koopmans Theorem (Phelps, 1965*b*), that such programs are dynamically inefficient. In such a case, efficiency will require the government to be a long-run net debtor ($g^* > 0$). Thus, in this case, given the private sector's saving propensity s, there would be no rate of time preference $\delta > 0$ consistent with a long-run zero debt, $g^* = 0$.[17]

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[17] Indeed, there would be no dynamically efficient *infinite* program consistent with

$$\lim_{t \to \infty} g(t) = 0.$$

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