"Hamiltonians"
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Hamiltonians. The laws of motion for a perfect-foresight economy, whether centrally planned or competitive, can be described by a Hamiltonian dynamical system or by a simple perturbation thereof. The Hamiltonian dynamical system and the Hamiltonian function which generates it are named for their inventor, the great Irish mathematician William Rowan Hamilton (1805–1865).

Hamilton's differential equations serve as the basic mathematical tool of classical particle mechanics (including celestial mechanics). Let $x(t) = (x_1(t), \ldots, x_i(t), \ldots, x_m(t))$ and $y(t) = (y_1(t), \ldots, y_i(t), \ldots, y_m(t))$ be m-vectors dependent on time t. Let H be a continuous, differentiable function of x, y, and t, H: $R^m \times R^m \times R \to R$. Think of H as the Hamiltonian function (HF) which generates Hamilton's differential equations,

$$\mathrm{d}x_i(t)/\mathrm{d}t = -\partial H(x(t), y(t), t)/\partial y_i(t)$$

and

$$\mathrm{d}y_i(t)/\mathrm{d}t = \partial H(x(t), y(t), t)/\partial x_i(t)$$

for $i = 1, \ldots, m$. If the Hamiltonian function H depends on time only through the variables $x(t)$ and $y(t)$, i.e., $\partial H/\partial t \equiv 0$, then the corresponding Hamiltonian dynamical system (HDS) is said to be *autonomous*. These differential equations are frequently interpreted in physics as solutions to some extremization problem. In mechanics for example, HDS is implied by the principle of least action. Since economic planning and many other economic problems involve maximization or minimization over time, it is unsurprising that the Hamiltonian formalism has substantial application in economics. Its appeal to economists goes much further than this. There is a duality (conjugacy, in the language of mechanics) between $x_i(t)$ and $y_i(t)$ which allows us to interpret one as a (primal) economic flow and the other as a (dual) economic price. Given this point of view, the Hamiltonian function (HF) itself has important economic interpretations. Hamiltonian dynamics not only arises in economic optimization problems but it also arises in descriptive economic models in which there is perfect foresight about asset prices. Hamiltonian dynamics applies in discrete time as well as in continuous time. In discrete time, the system of differential equations is replaced by a closely related system of difference equations. The right side of the equations describing Hamilton's law of motion need not be single-valued. The theory accommodates differential correspondences or difference correspondences, which naturally arise in economics.

Consider first the application of Hamiltonian approach to the theory of economic growth; see, e.g. the Cass–Shell (1976a) volume. A large class of economic growth models can be described by simple laws of motion based on the instantaneous production set T, with feasible production satisfying.

$$(c, z, -k, -l) \in T \subset \{(c, z, -k, -l) | (c, k, l) \geqslant 0\},$$

where c denotes the vector of consumption-goods outputs, z the vector of net investment-goods outputs, k the vector of capital-goods inputs, and l the vector of primary-goods inputs. There is an equivalent representation of static technological opportunities that is better suited to dynamic analysis: the representation of the static technology by its Hamiltonian function H.

Let p be the vector of consumption-goods prices and q be the vector of investment-goods prices. Define the Hamiltonian function $H(p, q, k, l)$ by

$$H(p, q, k, l) = \max_{(c', z')} \{pc' + qz' | (c', z', -k, -l) \in T\},$$

H is defined on the non-negative orthant and can be interpreted as the maximized value of net national product at the output prices (p, q) given input endowments (k, l).

Obviously, if we know the set T, then we know precisely the function H. If T is closed, convex, and permits free disposal, then H is continuous, convex and homogeneous of degree one in the output prices (p, q), and concave in the input stocks (k, l). If H is a function of (p, q, k, l) which is continuous, convex and homogeneous of degree one in (p, q), and concave in (k, l), then H corresponds to a unique T among closed, convex technologies permitting free disposal. In many dynamic applications, it is only the H representation which matters. Relax, for example, the free-disposal assumptions on T. For a given function H, the set T might not be unique, but the dynamics would be independent of the particular set T which generated the function H. Relax, as another example, the assumption that T is convex. Given an H which is convex in (p, q) and concave in (k, l), the set T will not be unique, but the continuous dynamics (HDS) will not be altered in an essential way.

Representation of the static technology by the Hamiltonian function permits one to describe the economic laws of motion as a Hamiltonian Dynamical System. In continuous time, the motion is described by

$$\dot{k}(t) \in \partial H(p(t), q(t), k(t), l(t))/\partial q(t)$$

(HDS)

$$\dot{q}(t) \in -\partial H(p(t), q(t), k(t), l(t))/\partial k(t)$$

where $\dot{k}(t)$ and $\dot{q}(t)$ are vectors of time derivatives and $(\partial H/\partial q)$ and $(\partial H/\partial k)$ are gradients (derivatives when H is differentiable). The first line of (HDS) is immediate from the definition of net investment since it reduces to $\dot{k}(t) = z(t)$, where $z(t)$ is the vector of net investment. The second line is an equal-asset-return condition which reduces to $\dot{q}(t) + r(t) = 0$, where $r(t)$ is the dual vector of shadow rental rates.

For discrete time, the Hamiltonian dynamical system is

$$k_{t+1} \in k_t + \partial H(p_t, q_t, k_t, l_t)/\partial q_t$$

(HDS)′

$$q_{t+1} \in q_t - \partial H(p_{t+1}, q_{t+1}, k_{t+1}, l_{t+1})/\partial k_{t+1}.$$

Line 1 is equivalent to $k_{t+1} = k_t + z_t$ and line 2 is equivalent to $q_{t+1} - q_t - r_{t+1} = 0$, where z_t is the time-t gross investment vector and r_{t+1} is the dual vector of shadow capital-goods rental rates in period $(t+1)$.

For openers, let us analyse the case where H is autonomous. This occurs if $p(t) = \bar{p}$ and $l(t) = \bar{l}$ for (HDS) or $p_t = \bar{p}$ and $l_t = \bar{l}$ for (HDS)′. Let (q^*, k^*) be a rest point to (HDS) or (HDS)′. Hence, we have

$$0 \in \partial H(\bar{p}, q^*, k^*, \bar{l})/\partial q,$$
$$0 \in \partial H(\bar{p}, q^*, k^*, \bar{l})/\partial k.$$

Consider the linear approximations about (q^*, k^*) of (HDS) and (HDS)′ (taken, for example, as if H were quadratic). Study the characteristic roots to the linearized systems. A simple but remarkable theorem due to Poincaré tells us that if λ is a root for the linearized, *autonomous* version of (HDS) then so is $-\lambda$. For the linearized, *autonomous* version of (HDS)′, we have if λ is a root, then so also is $1/\lambda$. If for (HDS), we could rule out pure imaginaries (Re $\lambda \neq 0$), then we would have: The dimension of the manifold in (q, k)-space of solutions tending to (q^*, k^*) as $t \to \infty$ is equal to the dimension of the manifold of solutions tending to (q^*, k^*) as $t \to -\infty$. This is the *saddle-point property*, where the manifold of forward solutions and the manifold of backward solutions each have dimension equal to half the total dimension of the space. Similarly, we would have the saddle-point property for (HDS)′, if the modulus $|\lambda|$ is unequal to unity.

Poincaré's result nearly gives us the saddle-point property. In the autonomous cases, the saddle-point property can be assured if the geometry of the Hamilton function is correct. We need to add very little to the convexity-concavity assumption (see Cass and Shell (1976b) and Rockafellar (1976)). Strict convexity in q and strict concavity in k will do the trick. So will a weaker uniform Hamiltonian steepness condition, which reduces to a value-loss condition; see, e.g., McKenzie (1968), and Cass–Shell (1976b).

What about non-autonomous systems, such as optimal economic growth with the constant, positive discount rate ρ? Here $c(t)$ or c_t is a scalar called felicity and usually denoted in optimal-growth problems by $u(t)$ or u_t. In this case, present prices must satisfy

$$-\dot{p}(t)/p(t) = \rho$$

or

$$-(p_t - p_{t-1})/p_t = \rho.$$

For simplicity, allow only for a single fixed factor and adopt the convention $l(t) = 1$, or $l_t = 1$.

It is natural then to re-express the systems (HDS) and (HDS)′ in terms of current prices $Q \equiv q/p$, rather than in terms of present prices q. We then have

$$\dot{k} \in \partial H(Q, k)/\partial Q$$

(PHDS)

$$\dot{Q} \in -\partial H(Q, k)/\partial k + \rho Q$$

and

$$k_{t+1} \in k_t + \partial H(Q_t, k_t)/\partial Q_t$$

(PHDS)′

$$Q_{t+1} \in Q_t - \partial H(Q_{t+1}, k_{t+1})\partial k_{t+1} + \rho Q_t.$$

The systems (PHDS) and (PHDS)′ are *perturbed* Hamiltonian dynamical systems. We no longer have Poincaré's root-splitting theorems in pure form: the roots split but not about 0 for (HDS) nor 1 for (HDS)′. The trick here is to strengthen the geometry of H to give a saddle-point property or something like it.

This is the basics of the approach taken by Cass and Shell (1976b), Rockafellar (1976) and Brock and Scheinkman (1976). Conditions are found on H which assure that either (PHDS) or (PHDS)′ along with transversality conditions defines a globally stable system. It suffices to strengthen the convexity-concavity of H by an amount dependent on ρ or (weaker) to strengthen the steepness of H by an amount dependent on ρ. (The Lyapunov function which does the trick is $V = (Q - Q^*) \cdot (k - k^*)$ in the continuous-time model.)

The Hamiltonian approach *through the Hamiltonian function* has proved remarkably successful in establishing sufficient conditions for the saddle-point property and related stability questions in a class of optimal economic growth models. The parallel programme of using the Hamiltonian formalism in optimal-growth theory to yield sufficient conditions for cycling or other dynamic configurations has not yet been pursued in a systematic fashion but should prove equally successful when applied. The success of the Hamiltonian approach in decentralized and descriptive growth theory has so far been very limited; see, Cass and Shell (1976b, Section 4). This has been disappointing. I still hope to see the Hamiltonian approach playing a pivotal technical role in, say, the dynamical analysis of overlapping-generations models, but there has not been much tangible encouragement for this hope.

Many of us first met Hamiltonian dynamical systems as necessary conditions for intertemporal maximization in the form of Pontryagin's Maximum Principle; see Pontryagin et al. (1962). See Shell (1967) for applications to economics and references.

Samuelson and Solow (1956) were probably the first in economics to mention the Hamiltonian formalism. For some of the history of Hamiltonian dynamics in economics, mathematics, and physics, and for some of the classical references see Magill (1970).

KARL SHELL

See also OPTIMAL CONTROL AND ECONOMIC DYNAMICS; SADDLEPOINTS.

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