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SELECTED ELEMENTARY TOPICS IN THE THEORY OF ECONOMIC DECISION-MAKING UNDER UNCERTAINTY*

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In order to illustrate the (partial equilibrium) theory of consumer behavior toward risk, I begin with an exposition of the Arrow-Pratt theory of risk aversion.[1] This theory can be extended quite easily to include a special three-asset (money, short-term government bonds, and shares of equity) model. The three-asset model, which is based on unpublished joint research of Albert Ando and myself, can be used for developing some fundamental propositions in monetary theory. I then proceed to the analysis of firm behavior under uncertainty. We must consider, in analyzing the goals of firm management, the general equilibrium setting of the firm in both real and financial markets; the notion of 'contingent-commodity' is presented in this context. The lectures conclude with remarks on the applicability of mean-variance analysis and market-line theories to 'large' economies.

For present purposes, let us accept the Von Neumann-Morgenstern axioms which lead to the expected-utility-maximization doctrine.[2] Let there be n possible states of nature, $s_1, \ldots, s_i, \ldots, s_n$, which an individual believes will occur with respective probabilities, $p_1, \ldots, p_i, \ldots, p_n; p_i \geqslant 0$ ($i = 1, \ldots, n$) and

$$\sum_{i=1}^{n} p_i = 1.$$

The individual's behavior can then be described as choosing an instrument z (say, an m-dimensional vector) from some decision set Z (say, a subspace of E^m) so as to maximize

$$\sum_{i=1}^{i=n} p_i U(z; s_i),$$

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[1] In preparing this lecture, I owe enormous debt to Kenneth Arrow [1]. See, esp., ch. 3. I also had access to a round-table survey prepared by Alvin Klevorick for the Boulder Meeting of the Econometric Society, August, 1971. My colleague, Stephen Ross, provided useful comments.

[2] See, e.g., Arrow [1], chapter 2.

where $U(z; s_i)$ is utility when state of nature s_i occurs and decision z has been taken. For simplicity, assume a static (or one-period) model in which ex-post (after Nature's di is cast) utility depends solely upon ex-post wealth, W, and let $U(W)$ denote utility of wealth, or end-period consumption. If wealth, or end-period consumption, is always desirable, then its marginal utility is positive,

$$U'(W) > 0 \text{ for } 0 < W < \infty.$$

Professor Arrow takes the view that the generalized St. Petersburg paradox implies that a proper specification includes the assumption that

$$\lim_{W \to 0} U(W) \text{ and } \lim_{W \to \infty} U(W)$$

exist and are bounded. The boundedness argument has never been convincing to me, but we shall have no need to use that assumption here. Purchase of fire insurance and the like, on the other hand, does suggest that individuals are willing to exchange (at fair odds or even somewhat less favorable than fair odds) 'risky wealth' for 'certain wealth'. Aberrations such as gambling aside, this suggests the specification of a strictly concave utility function; i.e., marginal utility declines with wealth, or

$$U''(W) < 0 \text{ for } 0 < W < \infty.$$

Individuals possessing utility functions with negative second derivatives are said to be *risk averse.* If $U''(W) = 0$, the individual is risk neutral. If $U''(W) > 0$, the individual prefers risk.

Arrow offers *two basic measures of risk aversion:*

$$\textit{Absolute} \text{ risk aversion: } \eta_A(W) = -U''(W)/U'(W)$$

$$\text{and } \textit{Relative} \text{ risk aversion: } \eta_R(W) = -WU''(W)/U'(W).$$

Remember that from Von Neumann-Morgenstern theory the utility function, $U(\cdot)$, is defined only up to positive linear transformation. Multiplying $U(W)$ by a positive constant also multiplies $U'(W)$ by a positive constant. The numerical value of $U''(W)$ is then of no significance, although its sign is of fundamental importance. Notice that addition of a positive constant to $U(W)$ has no effect on $U'(W)$ or $U''(W)$ and that multiplying $U(W)$ by a positive constant has no affect on the ratio $U'(W)/U''(W)$. Therefore, $\eta_A(W)$ and $\eta_R(W)$ are sensible measures of risk aversion.

Following Arrow and Pratt, we can study the application of Bernoulli-Von-

Neumann-Morgenstern theory to the choice between a risky asset (say, shares of equity) and a riskless asset (say, 'money'). It should be noted that in the intertemporal model where the individual simulultaneously makes portfolio decisions and consumption-saving decisions, it is unlikely that money can serve as the riskless asset – if indeed any asset can serve that role.

Define

X: random variable denoting rate of return on risky asset;

A: initial wealth;

a: investment in risky asset;

$m = A - a$: investment in riskless asset;

W: final (*ex-post*) wealth.

Then

$$W = A + aX. \tag{1}$$

The mathematical problem is to perform the following optimization:

$$\max_{a \in [0, A]} E[U(A + aX)], \tag{2}$$

where $E[\cdot]$ is the expectation operator. The above formulation assumes that the riskless asset bears no return. Expression (2) can be generalized by interpreting A as the *ex-post* final wealth if the individual specializes to cash, $a = 0$.

In practice corner solutions, $a = 0$ or $a = A$, are of great interest but for convenience, let us restrict ourselves to interior maxima, where from (2),

$$E[XU'(A + aX)] = 0. \tag{3}$$

We want to investigate the behavior of risky investment with respect to initial wealth. Differentiating in (3) yields

$$\frac{\mathrm{d}a}{\mathrm{d}A} = -\frac{E[U''(W)X]}{E[U''(W)X^2]}. \tag{4}$$

The denominator of the RHS of (4) is obviously negative, so we can write

$$\operatorname{sgn}(\mathrm{d}a/\mathrm{d}A) = \operatorname{sgn}\{E[U''(W)X]\}. \tag{5}$$

Note that if absolute risk aversion, $\eta_A(W)$, is decreasing in W, then

$$\eta_A(A + aX) \lesseqgtr \eta_A(A) \text{ for } \quad X \gtrless 0.$$

By definition of $\eta_A(\cdot)$,

$$U''(A + aX) \gtreqless -\eta_A(A)U'(A + aX) \text{ as } X \gtreqless 0.$$

Therefore, multiplication by X yields

$$XU''(A + aX) \geqslant -X\eta_A(A)U'(A + aX).$$

Taking expectations, yields

$$E[U''(A + aX)X] \geqslant -\eta_A(A)E[U'(A + aX)X] = 0,$$

from (3). Hence from (5),

$$(\mathrm{d}a/\mathrm{d}A) \geqslant 0 \text{ if } \eta_A(\cdot) \text{ is decreasing in } W. \tag{6}$$

This is a reasonable result: with decreasing absolute risk aversion, risky investment is not an inferior good. Arrow also calculates, e_{mA}, the elasticity of demand for cash balances

$$e_{mA} = \frac{\mathrm{d}\log m}{\mathrm{d}\log A}.$$

Since $m = A - a$,

$$e_{mA} - 1 = \frac{A}{m}\frac{\mathrm{d}m}{\mathrm{d}A} - 1 = \frac{A}{m}\left(1 - \frac{\mathrm{d}a}{\mathrm{d}A}\right) - 1$$

$$= \frac{(A-m) - A(\mathrm{d}a/\mathrm{d}A)}{m}\ .$$

But from (4) the above is equal to

$$\frac{(A-m)E[U''(W)X^2] + AE[U''(W)X]}{mE[U''(W)X^2]}.$$

Putting $m = A - a$, we get

$$\frac{E[U''(W)(aX^2 + AX)]}{mE[U''(W)X^2]} = \frac{E[X(a + AX)U''(W)]}{mE[U''(W)X^2]}\ .$$

Rewriting yields

$$e_{mA} = 1 + \frac{E[XWU''(W)]}{mE[U''(W)X^2]}\ . \tag{7}$$

Therefore,

$$e_{mA} \geqslant 1 \text{ iff } E\,[XWU''(W)] \leqslant 0. \tag{8}$$

From (8) it is straightforward to show that if relative risk aversion, $\eta_R(\cdot)$, increases with W, then $e_{mA} \geqslant 1$. Arrow points out that this is in accord with empirical studies of demand for money. Monetary theorists, however, probably place a greater emphasis on transactions demand rather than this liquidity preference component. I shall have more to say on this when the Ando-Shell three-asset model is presented, but before that I shall pursue some further implications of Arrow-Pratt theory.

One useful exercise is to consider the effects of shifts in the distribution X. Let θ be a shift parameter and denote the dependence of X upon θ by $X(\theta)$. Rewrite (3) as

$$E\{X(\theta)U'[A + aX(\theta)]\} = 0. \tag{9}$$

Differentiate with respect to θ,

$$E\{U''(W)[a(\mathrm{d}X/\mathrm{d}\theta) + X(\theta)(\mathrm{d}a/\mathrm{d}\theta)]\,X(\theta)$$
$$+ U'(W)(\mathrm{d}X/\mathrm{d}\theta)\} = 0$$

or

$$E\{[aU''(W)X(\theta) + U'(W)]\,(\mathrm{d}X/\mathrm{d}\theta)\}$$
$$+ (\mathrm{d}a/\mathrm{d}\theta)E\{U''(W)[X(\theta)]^2\} = 0. \tag{10}$$

From (10),

$$\operatorname{sgn}(\mathrm{d}a/\mathrm{d}\theta) = \operatorname{sgn} E(\{[aU''(W)X(\theta) + U'(W)]\,(\mathrm{d}X/\mathrm{d}\theta)\}). \tag{11}$$

Consider first, a shift of the form

$$X(\theta) = X + \theta \text{ so } \mathrm{d}X/\mathrm{d}\theta = 1. \tag{12}$$

$EU' > 0$, sgn $(EaXU'')$ = sgn (EXU'') and is therefore by (5) equal to sgn $(\mathrm{d}a/\mathrm{d}A)$. Therefore, demand for the risky asset increases with θ if $(\mathrm{d}a/\mathrm{d}A) > 0$.

Next consider shift of form

$$X(\theta) = (1 + \theta)X \text{ so } \mathrm{d}X/\mathrm{d}\theta = X. \tag{13}$$

For $\theta = -t$, this is the case of an income tax with complete loss offset (tax rate: $t > 0$). From (11)

$$\operatorname{sgn}(\mathrm{d}a/\mathrm{d}\theta) = \operatorname{sgn}\{EaX^2U'' + EXU'\}. \tag{14}$$

The second term on the RHS of (14) is zero by first-order condition (3). Therefore $\mathrm{d}a/\mathrm{d}\theta < 0$ or $(\mathrm{d}a/\mathrm{d}t) > 0$ (tax with loss-offset). In general, it can easily be shown that

$$a(\theta) = \frac{a}{1+\theta} \text{ or } a(t) = \frac{a(\text{zero tax})}{1-t}. \tag{15}$$

Problem for students: Rework expression (15) for case where there is no loss-offset. Calculation will not be so neat.

I now turn to the 'macroeconomic' three-asset extension that I promised earlier [3]. Initial wealth, A, is given. Individuals distribute their initial wealth among risky assets (equities and long-term bonds), C, short-term government bonds (or time deposits), B, and money (cash and demand deposits), M.

$$C + B + M = A. \tag{16}$$

Define

ρ: a random variable, rate of return on risky asset;
r: rate of return on bonds (*nonstochastic* for short-term governments);
ν: a random variable, rate of general price inflation;
$T(M)$: 'transaction cost' which decreases as money holdings increase, $T'(M) < 0$, more on this later;
W: a random variable, end-period real wealth.

We assume – as a *first-order approximation* – that intra-period expenditure is given independent of economic variables, but that the cost of transacting, $T(\cdot)$, depends on the 'average level of intraperiod money balances'. An inventory-theoretic story can be told *á la* Baumol and Tobin yielding the usual saw-tooth profile of money balances through the period. With care, we can conceptually define M as 'effective average money balance'.

[3] This presentation here is a highly condensed version of the Ando-Shell argument. This footnote is included to protect Ando.

Further define:

$$c = C/A \quad b = B/A \quad \text{and } m = M/A,$$

so that

$$c + b + m = 1. \tag{17}$$

End period wealth – available for consumption, or for consumption and saving in more complicated models – is

$$W = cA(1 + \rho - \nu) + bA(1 + r - \nu) + mA(1 - \nu) - T(mA). \tag{18}$$

The problem is to study interior maxima to $\Phi = EU(W)$ subject to eq. (17). Substitute $1-c-b$ for m in (18); the first-order condition $\partial\Phi/\partial b = 0$ can be written as

$$E\{U'(W)A\,[(1+r-\nu) - (1-\nu) + T'((1-c-b)A)]\} = 0$$

or

$$E\{U'(W)A\,[r+T'(M)]\} = 0. \tag{19}$$

But r and M are non-stochastic, so the LHS of (19) can be factored,

$$E[U'(W)]\,A\,[r+T'(M)] = 0. \tag{20}$$

Since initial wealth and marginal utility are positive, first-order condition (20) implies that

$$r = -T'(M). \tag{21}$$

Therefore the short-rate determines (uniquely if T'' is of one sign) holdings of cash balances, M.

There is another first-order condition to contend with. Setting $\partial\Phi/\partial c = 0$ yields

$$E\{U'(W)A\,[(\nu-1) + (1+\rho-\nu) + T'((1-c-b)A)]\} = 0. \tag{22}$$

Eq. (22) can be rewritten with the help of condition (21) as

$$E\{U'(W)A(\rho-r)\} = 0, \tag{23}$$

where, of course, $E\rho > r$ if $U'' < 0$ and all three assets are held. Notice that condition (23) can be used in applying Arrow-Pratt theory to the extended model. While from (21) M is independent of the distribution ρ, (23) yields important comparative statics theorems about risky-asset demand. In Arrow's terminology, set

$$X = A(\rho-r)$$

and apply to the extended model all those Arrow-Pratt theorems that are a consequence of expression (5).

I have to admit that our transactions cost function, $T(M)$, is a very unspecific black-box, although it has respectable antecedents in the Baumol-Tobin square-root formula. In general, $T(\cdot)$ should depend on expenditure which would be a random variable dependent upon general consumption and investment opportunities. If, however, these qualifications can be accepted as of 'second order', then the lecture has demonstrated that the *demand for money is independent of all rates of return except on that asset that dominates money* for portfolio purposes ('short-term government bonds'). Also, *given the supply of money, the short-term interest rate is determined independently of expectations of price level inflation and independently of the supply of Treasury bills* (short-term government bonds).

Production decisions under uncertainty and the general-equilibrium theory of the competitive firm. If individuals maximize expected utilities based on subjective probabilities, what goals should the competitive firm pursue in a risky environment? Inventory theorists have based their arguments on the expected profit-maximization hypothesis – although, to my knowledge, the question of where the probabilities come from was never posed. Recent writers on firm behavior have generalized the goal to that of maximizing expected 'utility' of profits, i.e., maximizing the expected value of some concave function of profits.

On the other hand, the basic general-equilibrium theory of the firm (static, dynamic, risky, or whatever) posits that firms seek to maximize current value or equivalently maximize the value on outstanding equity shares. Managers follow this rule in order (1) maximize the value on their own equity shares, (2) avoid shareholder rebellion, and most plausibly (3) avoid takeover.

In order to pursue the consequences of the 'general-equilibrium share-price maximization model', consider a firm that 'produces' corn. There are two possible states of nature: (1) rain and (2) drought. The manager of the firm believes the probability of rain to be p and of drought to be $1 - p \cdot c_R$ is corn output if rain; c_D is corn output if drought. Let me pretend that if corn is planted very deep, then it flourishes in rain. If planted very shallow it flourishes in drought.

Turn to figure 1. If the manager plants deep, he assures himself of OA corn production of rain, but zero production if drought. If he plants shallow, production is OB if drought, zero if rain. By planting some of his fields shallow and some deep, he could achieve any point on the line segment AB (shown dotted). There may be advantages to planting at intermediate depths allowing for a curved production possibility frontier (the solid curve AB). (The pie-shaped production set is the boundary and interior of ABO.)

If the manager produces for his own comsumption, in the absence of trade he would choose a point on the PPF (AB) that maximizes

$$pU(c_R) + (1-p)U(c_D).$$

The dashed indifference curve represents the locus of points

$$pU(c_R) + (1-p)U(c_D) = \beta \text{ (a constant)}.$$

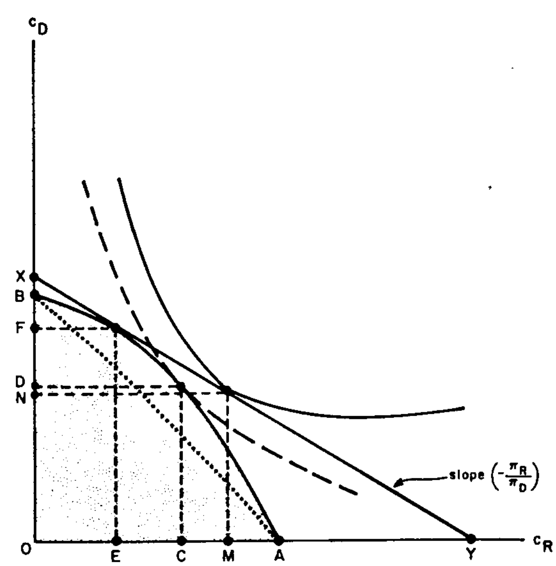

Fig. 1.

In autarchy, the manager's optimal plan is to plant his field so as to harvest (and consume) *OC* if rain and *OD* if drought.

For the general equilibrium theorist, a perfect insurance market is one where all contingent commodity markets exist and are perfect [4]. There are two contingent commodities from our little example that 'span the states of nature': (1) Deliver one bushel corn if rain, zero if drought. (2) Deliver one bushel corn if drought, zero if rain. Assume that the respective prices of these contingent claims are π_R and π_D.

To maximize the value of the firm, the manager plants his fields to make the PPF *AB* tangent to a line with slope $(-\pi_R/\pi_D)$. If rain he produces only *OE*, but if drought his production is increased to *OF*. Even if the manager is the owner, the value maximizing production plan leads to utility maximization. By trading at market contingent claims prices his opportunity set is expanded; it is the triangle *OXY*. Benefitting from trade, his utility is increased. A higher indifference curve (drawn in solid) defined by

$$pU(c_R) + (1-p)U(c_D) = \beta' > \beta,$$

is tangent to the hypotenuse of his (triangular) opportunity set *OXY*. Consumption is *OM* if rain, *ON* if drought. Note that production is at (*OE*, *OF*) no matter what the manager's tastes and probability beliefs are. The prices π_R and π_D 'code the market's beliefs and tastes'.

How is this theory related to the expected profit maximization doctrine? Obviously, the two yield the same results if and only if

$$p/(1-p) = \pi_R/\pi_D.$$

This is not likely to be the general case when individuals differ as to probability beliefs *or* if $U(\cdot)$ is a strictly concave function.

Concluding remarks on mean-variance analysis

In several of the other lectures in this volume, analysis will be based on the Markowitz-Tobin (E, V) or (μ, σ) model. It may be worth your while to attempt to relate that material to the theories presented in this lecture.

How is (E, V) analysis related to the doctrine of expected-utility maximization? The (E, V) analysis can be thought of as a special case where either (1) the Von Neumann-Morgenstern utility is quadratic (thus possessing in the large objec-

[4] This requirement may be too strong. Professor Stiglitz's lectures on separation into mutual funds will examine cases when fewer contingent claims markets are necessary.

tionable negative marginal utility) and/or (2) the underlying probability distribution is normal and is thus fully described by the two parimeters μ and σ.

What about the tangency of the market line with the efficient μ–σ frontier – wich implies an optimal portfolio diversification which is independent of investor wealth, etc.? This argument is, of course, a special case of our general-equilibrium contingent-claim model described in figure 1.

What are the advantages of the (E, V) model over the more general theory presented here? I must say that I can see none for theoretical analysis. But the (E, V) model has been around, is well understood, and relatively well-suited to computation.

Is there then any sense in which the mean-variance model can be argued to approximate the behavior of a 'large economy'? Can the Central Limit Theorem be of any use here? The answer seems to be yes. My colleague Stephen Ross [3] has shown that in a single-factor market model with a large number of securities, individual portfolio behavior is unaltered, and the equilibrium remains intact for general factor models. In particular, as long as the degree of dependence between anticipated asset returns is not too high, the security-line equation of the capital-market model will become an increasingly good approximation as the number of securities becomes large.

References

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