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The Structure and Stability of Competitive Dynamical Systems

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1. INTRODUCTION AND SUMMARY

The theory of economic growth has itself developed at an enormous rate over the past twenty years. A careful reading of the major work in this extensive literature[1] leaves one quite definite impression: The analysis of the long-run behavior of idealized economies has been very uneven. On the one hand, quite general results are available concerning the theory of maximal growth, broadly defined to include the problem of production-maximal growth, as well as the closely related problem of consumption-optimal growth with zero (net) discounting. On the other hand, only a few simple examples and special cases have been examined in the theory of optimal growth with positive discounting, or of descriptive growth with linear saving–investment hypotheses.

In this paper we provide a general framework which encompasses these as well as many other problems in the theory of economic growth, or more broadly, the theory of economic dynamics. Our approach is to describe competitive dynamics as Hamiltonian dynamics, where the Hamiltonian can be written as a function of *present* output prices and current input stocks, and can be interpreted as the present value of net national product (equal, by duality, to the present value of net national income). Such a Hamiltonian dynamical system is competitive in the sense that it derives from the perfect-foresight, zero-profit, asset-market clearing equations

[1] Our references contain a large sample of the basic work concerned with global stability in economic growth, ranging from the early work on the "turnpike" theorem (e.g., [9, 18, 20, 23]), through the continuing work on optimal growth (e.g., [2, 11, 19, 24] for zero discounting, and [5, 15, 28, 35] for positive discounting) and on descriptive growth (e.g., [14, 32, 34, 36]), to the recent work on the "Hahn problem" (e.g., [4, 8, 12, 33]). We have not attempted to include systematic references to closely related topics, for example, local stability analysis.

arising in descriptive growth theory, and is consistent with (i) efficiency pricing conditions developed in the Malinvaud [17] tradition, and (ii) Euler's conditions or, more generally, Pontryagin's maximum principle (in either their usual continuous-time or their analogous discrete-time formulations), applying to production-maximal or consumption-optimal growth problems.

The problems we are interested in naturally lead to cases in which the Hamiltonian function is convex in present prices (the costate variables) and concave in current stocks (the state variables). Indeed, every convex technology is essentially characterized by such a convex–concave Hamiltonian function, and every convex–concave Hamiltonian function essentially represents a unique convex technology (see, for instance, the companion analysis of Lau [16], and related exposition of Cass [7]). Since our analysis originates in this duality, it is not surprising that we find the geometry of the Hamiltonian function to be the fundamental determinant of the long-run behavior of a competitive dynamical system.

It is customary and convenient in studying the dynamics of optimal growth to cast the analysis in terms of *current* output prices rather than present output prices. In the case with zero discounting, a (golden rule path) stationary point is also a saddlepoint of the current value Hamiltonian function. Thus, a simple assumption which insures uniqueness of the stationary input stocks is that the Hamiltonian be strictly concave in current input stocks. This strict concavity condition is unnecessarily strong. An assumption of "real Hamiltonian steepness" at the stationary point suffices to insure uniqueness. Moreover, a uniform strengthening of this steepness condition also allows us to establish convergence of the optimal path to the stationary input stocks. This stability analysis for consumption-optimal growth with zero discounting can be easily extended to apply to the case of production-maximal growth. In the latter case, our strengthened assumption of "real Hamiltonian steepness" at the von Neumann ray insures convergence to that ray, and is equivalent to Radner's assumption of bounded value-loss.

For optimal growth with constant, positive discounting, a (modified golden rule path) stationary point is not a saddlepoint of the Hamiltonian function. Nevertheless, when our strengthened "real Hamiltonian steepness" condition is modified by a term which depends on the discount rate, we again establish uniqueness of and convergence to the stationary input stocks. Furthermore, even when the discount rate is only asymptotically constant, some additional tightening of this modified steepness condition still permits us to establish global stability.

Our steepness conditions lead very naturally to a particular choice of Lyapunov function for the stability arguments. This particular function

has been used previously by Samuelson [29] and Rockafellar [25] in analyzing the stability of consumption-optimal growth with zero discounting (and, under a liberal interpretation, by Radner [23] in analyzing the stability of production-maximal growth). Our extension of these earlier results provides both a coherent unification of the maximal growth models, as well as a powerful justification for our Hamiltonian approach to the analysis of competitive dynamical systems.

In the area of descriptive growth, the analysis of dynamical behavior has typically been limited to studying fairly specific examples. From our present perspective, the fundamental difficulty in obtaining definite results for more general models seems quite evident: Since the interest rate implicitly depends on the whole evolution of prices and quantities, there is no direct analog to our "real Hamiltonian steepness" conditions. Nonetheless, by exploiting the particular structure of descriptive growth under the Marxian saving–investment hypothesis, we are able to establish a broad class of circumstances in which similar conditions entail global stability. Thus, our Hamiltonian approach yields one of the first reasonably general results on the long-run performance of a decentralized, capital accumulation process. Extensions of this approach to cover a wider range of descriptive growth models appears to us an important and promising prospect.

We believe that our Hamiltonian approach has many further possibilities for application. In particular, even without complete convexity in the underlying model, the solutions to many other interesting optimal control problems must exhibit Hamiltonian dynamics of the sort we analyze in this paper, i.e., must obey dynamical laws derived from a convex–concave Hamiltonian function. Thus, for example, our results are immediately applicable to a fairly general version of the neoclassical investment model. On a broader tack, we also believe that our duality emphasis may be quite useful in investigating the various existence problems which are deliberately sidestepped in the present paper. For instance, the existence of solutions to the particular differential equations (or differential correspondences) we define as representing optimal growth would follow from a duality theorem for concave programming in some appropriate infinite-dimensional space. More speculatively, we think it may be possible to derive the solutions to particular descriptive growth models in a similar manner.

In either case, this casual conjecturing remains to be verified by further research, part of which we ourselves plan to undertake in the near future.

The extended Cambridge-versus-Cambridge debate has taught us that insights based on the aggregate production function are of only limited value for unifying models of heterogeneous capital accumulation. It is our hope that our Hamiltonian approach will, at least to some extent,

provide the unifying principle that Cambridge, Massachusetts has been seeking.

2. The General Model

A. *Technology and Market*

Let

consumption-goods output (or sometimes "utility output") = $(c_1, \ldots, c_\mu) = c$,
net investment-goods output = $(z_1, \ldots, z_\nu) = z$,
capital stocks = $(k_1, \ldots, k_\nu) = k$,
primary factors = $(l_1, \ldots, l_\xi) = l$, and
technology = $T \subset \{(c, z, -k, -l): (c, k, l) \geqslant 0\}$.

While the general tenor of our maintained assumptions about technology is quite conventional, there is some special structure which is either necessary or useful for accomplishing our specific goal of establishing global stability. Thus, while listing our maintained assumptions we also mix in comments about what purposes they serve (and what alternative assumptions would suffice).

Assumption (T1). Feasibility of nonnegative inputs. If $(k, l) \geqslant 0$, then there exists (c, z) such that $(c, z, -k, -l) \in T$.

This assumption simply states that, given a sign convention for measuring inputs, any input combination can be used. In conjunction with assumption (T8) below, it allows restriction of the Hamiltonian function to a simple domain of definition, the nonnegative orthant. This is merely convenient, and so (T1) could be replaced by a weaker regularity requirement concerning the projection of T onto the input space, for example, that this projection be closed.

Assumption (T2). Feasibility of boundary production. T is closed.

This assumption is a standard regularity requirement. Since so much of our analysis depends on it, directly or indirectly, it is difficult for us to see how it could be dispensed with.

Assumption (T3). Diminishing returns in production. T is convex.

While this assumption too is a standard regularity requirement, it is not indispensable. For instance, even with increasing returns to capital stock inputs, if production possibility sets are convex, then the technology is representable by a Hamiltonian function, and furthermore, an optimal

growth path is describable by a Hamiltonian dynamical system. (The last requires some additional regularity to replace convexity; such regularity is implicitly assumed, for example, when the technology is specified in terms of industry or sectoral production functions together with input stock adding-up constraints.) Though we will not put much stress on this aspect of our analysis, the potential for handling such phenomena as increasing returns is a substantial reason for devoting serious attention to Hamiltonian dynamics, as well as an important area in the future development of economic dynamics.

ASSUMPTION (T4). Productivity of the technology. There exists $(c, z, -k, -l) \in T$ such that $(c, z) > 0$.

This assumption introduces the potential for balanced growth, since it means that, with appropriate endowments of initial capital stocks and primary factors, capital stocks can be maintained forever. While such an assumption is a common feature of most growth models, it is not at all innocuous, because it completely rules out the phenomenon of depletable resources. However, developing a general theory of the behavior of prescriptive or descriptive growth models when there are depletable resources is a major task in itself, and not one we even attempt here.

ASSUMPTION (T5). Constant returns to scale. T is a cone.

This assumption is primarily a convenience for expositing our Hamiltonian approach, since, when actual technological processes exhibit diminishing returns to scale, it is tantamount to introducing a fictitious input which "earns" residual income or accounting profit. In other words, because any set can be conceived as the projection of a cross-section of a cone in one higher dimension, there is absolutely no loss of generality in conducting analysis, when it is useful, in terms of such a cone plus its restriction. Of course some care must be taken in providing economic interpretation of this maneuver (partly amplified by the discussion in [6, pp. 273–76]). Thus, in particular, the reader should beware of a literal reading of our treatment of the optimal growth model; in our specification of that model, the "primary factor" typically summarizes both the influence of exogenous factor availability, as well as the influence of diminishing marginal utility.

ASSUMPTION (T6). Necessity of primary factors: If $(c, z, -k, -l) \in T$ and $l = 0$, then $c = 0$ and $z_i \{\leqq\} 0$ according as $k_i \{\geqq\} 0$ for $i = 1, 2, ..., \nu$.

This assumption parallels the no-free-lunch postulate of general equilibrium analysis, but is stronger, since it requires that gross outputs be zero when primary factor inputs are zero, even when some capital stock

inputs are positive. The importance of the assumption is that, in conjunction with other of our assumptions, notably (T2), (T5) and (T7), it implies that too large capital stocks simply cannot be maintained over time. Because we also assume a bounded rate of depreciation (see (T7) immediately below), this upper bound on feasible capital stocks also provides an absolute bound on feasible produced outputs.[2] We will have need to refer to these implied restrictions later on in our analysis, and so at this point we somewhat prematurely formalize them in the following proposition (which implicitly appeals to both our maintained assumptions about technology and our special assumptions about exogenous factor availability; see (A1) and (A3) further below).

BOUNDEDNESS LEMMA. There is a finite bound $B < \infty$ such that every production point $(c, z, -k, -l) \in T$ which is replicable or feasible satisfies the boundedness restriction

$$\|(c, z, k)\| < B. \tag{B}$$

By way of further interpretation, it is interesting to observe that assumptions (T4) and (T6) play a role in our model analogous to that of the Inada conditions in the standard one-good model, and are crucial, for instance, in establishing the existence of stationary points which exhibit intertemporal consumption efficiency (modified golden rule paths).

ASSUMPTION (T7). Bounded rates of depreciation. There exists

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \lambda_\nu \end{bmatrix}$$

[2] These assertions obviously presuppose that primary factors are scarce, that is, are available in limited amounts. Their proof is relatively straightforward if we restrict attention to production points $(c, z, -k, -l) \in T$ which (i) employ a unit level of primary factors $\|l\| = 1$, and (ii) yield nonnegative investment goods output $z \geqslant 0$, or which are *replicable*. In this special case, the standard argument based on assumptions (T2), (T5), and (T6) establishes that the set

$$\{(c, z, k)\colon (c, z, -k, -l) \in T, z \geqslant 0 \text{ and } \|l\| = 1\}$$

is bounded. When we turn attention to production points which might be observed along a path achievable from given, finite initial capital stocks and primary factors, or, which are *feasible*, their proof is of explicitly dynamical character, as well as of significantly more complexity. The useful paper by Peleg and Ryder [22] contains a complete discussion for a particular discrete-time model. The related argument applying to our continuous- and discrete-time models is similar, though not identical.

with $\lambda_i \in (0, \infty)$ (continuous time) or $\lambda_i \in (0, 1]$ (discrete time) for $i = 1, 2, ..., \nu$ such that if $(c, z, -k, -l) \in T$, then $-\Lambda k \leqslant z$, and

ASSUMPTION (T8). Free disposal in allocation. If $(c, z, -k, -l) \in T$, $(c', z' + \Lambda(k' - k), -k', -l') \leqslant (c, z, -k, -l)$ and $(c', z') \geqslant (0, -\Lambda k')$, then $(c', z', -k', -l') \in T$.

Assumptions (T7) and (T8), taken together, are nothing more than an alternative statement of the standard assumptions of depreciation at constant (relative) rates plus free disposal of (gross) output and input. While there are several implications of these assumptions which are imporant for our analysis, here we will only emphasize one, that having to do with characterizing intertemporal consumption (or, more generally, net output) efficiency. Namely, (T8) (which only makes sense in light of (T7)) entails that output, both consumption-goods *and* investment-goods, must have associated nonnegative efficiency prices. (The reason for this is basically that (T8) means that having more capital stocks today does not require having less capital stocks, and, hence, less consumption-goods output, tomorrow.) This implication provides the rationale underlying our introduction of unlimited free disposal of output in the definition of the market M below, and is especially important in justifying our particular description of optimal growth as necessarily involving asymptotically zero or finite present value of capital stocks. While it seems likely that our results can be generalized to require somewhat less special structure than is imposed by (T7) and (T8) (for example, this is suggested by Rockafellar's [26] local analysis), the complete details of such extension remain to be seen.

Corresponding to the various commodities and the technology, define

output prices $= (p_1, ..., p_\mu, q_1, ..., q_\nu) = (p, q)$,
input rents $= (r_1, ..., r_\nu, w_1, ..., w_\xi) = (r, w)$, and
market $= M \subset \{(p, q, r, w): (p, q, w) \geqslant 0\}$.

M is dually related to T by

$$M = \{(p, q, r, w): pc' + qz' - rk - wl \leqslant 0 \quad \text{for all} \quad (c', z') \leqslant (c, z), (c, z, -k, -l) \in T\}, \quad (1)$$

and, as noted above, contains all the price configurations which might appear in any snapshot of a competitive dynamical system. Even without assumptions (T1)–(T8), M is a closed convex cone having the additional properties of nonnegativity of output prices, if $(p, q, r, w) \in M$, then $(p, q) \geqslant 0$, and free disposal in valuation, if $(p, q, r, w) \in M$, $(p', -r', -w') \leqslant (p, -r, -w)$ and $p' \geqslant 0$, then $(p', q', -r', -w') \in M$.

B. *Accumulation and Growth*

Initial capital stocks and primary factors are given exogenously. Hence, letting bars denote this fact, all feasible paths of real variables must satisfy, in continuous time, $t \in [0, \infty)$,

$$(c(t), z(t), -k(t), -l(t)) \in T, \dot{k}(t) = z(t), \quad k(0) = \bar{k} \text{ and } l(t) = \bar{l}(t), \quad (2)$$

or, in discrete time, $t = 0, 1, \ldots,$

$$(c_t, z_t, -k_t, -l_t) \in T, k_{t+1} = k_t + z_t, \quad k_0 = \bar{k} \text{ and } l_t = \bar{l}_t. \quad (2')$$

In (2) it is understood that, for example, $\{z_i(t): 0 \leqslant t < \infty\}$ is summable, or $\{k_i(t): k_i(t) = k_i(0) + \int_0^t z_i(s)\, ds$ and $0 \leqslant t < \infty\}$ is absolutely continuous on any finite interval $0 \leqslant t \leqslant t' < \infty$; a similar remark will apply to the corresponding price variables.

C. *Intertemporal Profit Maximization*

Parallel with (2) or (2′), all "feasible" paths of price variables must satisfy

$$(p(t), q(t), r(t), w(t)) \in M, \quad \dot{q}(t) = -r(t) \text{ and } q(0) \geqslant 0, \quad (3)$$

or

$$(p_t, q_t, r_t, w_t) \in M, \quad q_{t+1} = q_t - r_{t+1} \text{ and } q_{-1} \geqslant 0.^3 \quad (3')$$

We define paths satisfying, or solutions to (2) and (3) or (2′) and (3′), to be a *competitive dynamical system*, provided, in addition, prices are nontrivial, for some t $(p(t), q(t), r(t), w(t)) \neq 0$ or $(p_t, q_t, r_t, w_t) \neq 0$,[4] and there is static profit maximization,

$$p(t)\, c(t) + q(t)\, z(t) - r(t)\, k(t) - w(t)\, l(t) = 0, \quad (4)$$

[3] We note in passing that the notion of dual instability, prominent in the early literature concerning dynamic Leontief models, seems nothing more than a special case of the observation that (3) or (3′) can be essentially converted into (2) or (2′) by reversing the flow of time (e.g., considering $t \in [0, -\infty)$) and (ii) adding appropriate exogenous constraints (e.g., fixing $q(0) = \bar{q}$, $p(t) = \bar{p}(t)$).

It is worth stressing explicitly that, unlike feasible paths of real variables, "feasible" paths of price variables, that is, all paths affording at best zero profit opportunities, are not subject to exogenous constraints other than nonnegativity of initial investment goods prices (and this too is actually superfluous in the case of continuous time, given our particular definition of M).

[4] Paths of price variables which are identically zero are of little interest in either prescriptive or descriptive growth theory. In fact, the same is true of paths which are identically zero after some point in time; competitive dynamical systems with such prices correspond to finite-lived economics.

or

$$p_t c_t + q_t z_t - r_t k_t - w_t l_t = 0. \tag{4'}$$

Given the definition of M in (1), Eqs. (3)–(4) or (3′)–(4′) are nothing but the well-known perfect foresight, intertemporal profit maximization conditions.

D. *The Hamiltonian Representation*

Static profit maximization (at nonnegative output prices $(p, q) \geqslant 0$),

$$(c, z, -k, -l) \in T,$$

and

$$pc' + qz' - rk' - wl' \leqslant pc + qz - rk - wl = 0 \quad \text{for all} \quad (c', z', -k', -l') \in T, \tag{5}$$

is equivalent to the conditions that $(c, z, -k, -l)$ is an optimal solution to the concave programming program,

$$\text{maximize } pc' + qz' \quad \text{subject to} \quad (c', z', -k, -l) \in T. \tag{NNP}$$

(p, q, r, w) is an optimal solution to the (dual) convex programming problem,

$$\text{minimize} \quad r'k + w'l \quad \text{subject to} \quad (p, q, r', w') \in M, \tag{NNI}$$

and the values of NNP ($\equiv$ Net National Product) and NNI ($\equiv$ Net National Income) are equal,

$$\begin{aligned} \text{value of NNP} &= pc + qz \\ &= \text{value of NNI} \\ &= rk + wl. \end{aligned} \tag{6}$$

This equivalence suggests the potential usefulness of defining the *Hamiltonian function*

$$H(p, q, k, l) = \text{value of NNP}. \tag{7}$$

It is fairly straightforward to establish that H exhibits the following fundamental properties.

Property (H1). H is defined on the nonnegative orthant $\{(p, q, k, l): (p, q, k, l) \geqslant 0\}$.[5]

Property (H2). H is continuous in (p, q, k, l).

[5] Note that, though T is not compact, from assumptions (T2), (T6), and (T7), a cross-section like $\{(c', z', -k', -l'): (c', z', -k', -l') \in T \text{ and } (k', l') = (k, l)\}$ is compact. The parallel definition of H in terms of the value of NNI may run into difficulty when (k, l) is on the boundary of the projection of the technology set on the input space, i.e., when $(k, l) \not> 0$; see Footnote 7 below.

Property (H3). H is nondecreasing in p and l.

Property (H4). H is convex and linear homogeneous in (p, q), and concave and linear homogeneous in (k, l).

And, most important,

Property (H5). $\partial H(p, q, k, l)/\partial(p, q) = (c, z)$,[6] and, at least for $(k, l) > 0$,[7] $\partial H(p, q, k, l)/\partial(k, l) = (r, w)$.

Thus, the generalized gradients of H with respect to (p, q) are static profit-maximizing outputs (and conversely), while, for all practical purposes, the generalized gradients of H with respect to (k, l) are competitive input rents (and conversely).

Properties (H1)–(H5) follow directly from assumptions (T1)–(T7): Roughly speaking, for the purpose of analyzing static competitive or efficient allocation, every well-behaved technology set can be fully represented by a well-behaved Hamiltonian function.[8] Furthermore, given H, one can reconstruct T (if there is unlimited free disposal of output), or at least the critical "northeast" boundary of T (if there is not), completely.

[6] Here and after we use $\partial f(x, y)/\partial x$ to denote some (perhaps particular) or all generalized gradients of $f(x, y)$ with respect to x; e.g., when, for given y, the function $f(x, y)$ and its domain $D_x(y)$ are convex in x, some or all vectors $\partial f(x, y)/\partial x$ such that

$$f(x', y) - f(x, y) \geqslant (x' - x)\frac{\partial f(x, y)}{\partial x} \qquad \text{for all} \quad x' \in D_x(y).$$

[7] Some version of "constraint qualification"—here we use Slater's condition, the obvious choice for our problem—is required to insure that, in addition to (7),

$$H(p, q, k, l) = \text{value of NNI}.$$

(See the extended discussion of this point in [6, pp. 277–79]. We should emphasize that for the applications in this paper, whenever (5) obtains, (6) does also, so that by presumption boundary points present no difficulty (i.e., hypothesizing a solution to (2)–(4) or (2′)–(4′) already entails $\partial H(p, q, k, l)/\partial(k, l) = (r, w)$ without qualification).

[8] It is worth at least remarking on the relative merits of our Hamiltonian function representation vis-a-vis the more common production transformation representation, say, $F(c, z, k, l) = 0$, where F is defined implicitly by the "northeast" boundary of T. In general, the former has a distinct advantage over the latter: The domain of H is the fairly simple Cartesian product $\{(p, q): (p, q) \geqslant 0\} \times \{(k, l): (c, z, -k, -l) \in T\}$, while the domain of F is the more complicated boundary set $\{(c, z, k, l): (c, z, -k, -l) \in T$ and (c, z) is maximal in $\{(c', z'): (c', z', -k, -l) \in T\}\}$. In other words, in order to use H, one only needs to know all feasible input combinations, while in order to use F, one also needs to know all maximal output combinations, in effect, T itself. To our minds, this is a dominating superiority in most contexts where some functional representation of T is required.

This near equivalence between the T and H representations of convex[9] technologies is, as mentioned earlier, precisely stated and fully investigated in [7, 16].

For our objective here, it is more natural to represent the technology in terms of H rather than T. In particular, the equivalence between (5) and (6) together with the properties (H1)–(H5) mean that we can very succinctly rewrite the competitive dynamical system (2)–(4) or (2′)–(4′) as the *Hamiltonian dynamical system*

$$\begin{cases} \dot{k}(t) = \dfrac{\partial H(p(t), q(t), k(t), \bar{l}(t))}{\partial q}, & k(0) = \bar{k}, \\ \dot{q}(t) = -\dfrac{\partial H(p(t), q(t), k(t), \bar{l}(t))}{\partial k}, & q(0) \geqslant 0, \end{cases} \tag{8}$$

or

$$\begin{cases} k_{t+1} = k_t + \dfrac{\partial H(p_t, q_t, k_t, \bar{l}_t)}{\partial q}, & k_0 = \bar{k}, \\ q_{t+1} = q_t - \dfrac{\partial H(p_{t+1}, q_{t+1}, k_{t+1}, \bar{l}_{t+1})}{\partial k}, & q_{-1} \geqslant 0. \end{cases} \tag{8′}$$

The most striking feature of this formulation is its suggestion that *it is properties of the technology set as reflected in properties of the Hamiltonian function which are basic to the stability analysis of competitive dynamical systems*. This suggestion is amply borne out in the sequel. The representation (8) or (8′) just as obviously provides *a very convenient framework for constructing examples of competitive dynamical systems*, as we shall also clearly illustrate in the succeeding analysis.

E. *Simplifying Assumptions*

We now make some special assumptions in order to simplify the analysis that follows.

ASSUMPTION (A1). There is a single primary factor, $\xi = 1$, and it is unchanging in amount, $l(t) = \bar{l}(t) = 1$ or $l_t = \bar{l}_t = 1$, say. This assumption is made so that we can sensibly talk about stationary points. An essentially equivalent assumption would be that, however many, all primary factors grow asymptotically at the same constant rate (the "natural" growth rate).

[9] Since an output price system "convexifies" just the production possibility sets derived from a nonconvex technology, in that case one can only recover the union of the convex hulls of cross sections of the representation T, or its "northeast" boundary, from the representation H.

ASSUMPTION (A2). There is a single consumption good, $\mu = 1$. Since we can interpret output of this good as instantaneous utility, we are able to analyze most consumption-optimal growth problems treated in the literature. Here the single final utility output might be "produced" from a multi-dimensional vector of "intermediate" consumption-goods output.[10]

For descriptive growth theory, the assumption of a single consumption good is more restrictive, but we feel that inclusion of the general case would needlessly complicate out present analysis. Alternatively, we can simply duck this issue by postulating the existence of a sufficiently well-behaved consumption aggregate.

ASSUMPTION (A3). The initial endowment of each capital good is positive and finite $0 < \bar{k} < \infty$.

This assumption together with (T6) guarantees the existence of positive capital stocks for all time, $k(t) > 0$ or $k_t > 0$, a fact we use indirectly in analyzing optimal growth, directly in analyzing descriptive growth.

F. *Saving–Investment Behavior*

To close the Hamiltonian dynamical system, (8) or (8′), requires at a minimum, specification of initial prices of capital stocks, $q(0) = \bar{q}$ or $q_{-1} = \bar{q}$, and of dated prices of consumption-goods output, $p(t) = \bar{p}(t)$ or $p_t = \bar{p}_t$.

This may be accomplished directly or indirectly, depending on the particular problem under scrutiny. Consider, for example, the problem of optimal growth with constant, positive discounting. The maximand is

$$\int_0^\infty e^{-\rho t} c(t)\, dt, \tag{9}$$

or

$$\sum_{t=0}^{\infty} (1 + \rho)^{-t}\, c_t\,, \tag{9′}$$

where $c(t)$ or c_t is interpreted as utility output and ρ its associated time

[10] Even though T is assumed to be a cone, we could allow for diminishing returns in transforming "intermediate" consumption-goods output into the final utility output. Again, this follows from the fact that any convex set can be described as the projection of a cross-section of a convex cone of one higher dimension. In this situation, however, the interpretation of the fictitious input required for marginal productivity factor payment exhaustion of product is somewhat fanciful.

discount rate, the utility or "real" interest rate. Maximization of (9) or (9′) is constrained by technology and endowments (2) or (2′). For a feasible path to be optimal, it must be efficient. Thus, it is required that there exist nonnegative imputed prices, $(p(t), q(t)) \geqslant 0$ for all t or $(p_t, q_{t-1}) \geqslant 0$ for all t, such that the laws of motion, (8) or (8′), hold along an optimal path. But since, on an optimal path, the utility interest rate is assumed equal to ρ, it is also required that the price of utility output be declining at this rate, that is, that

$$-\frac{\dot{p}(t)}{p(t)} = \rho, \tag{10}$$

or

$$-\frac{(p_t - p_{t-1})}{p_t} = \rho. \tag{10′}$$

If we now choose initial utility as the numeraire commodity, $p(0) = 1$ or $p_0 = 1$, then (10) or (10′) becomes, in closed form,

$$p(t) = e^{-\rho t}, \tag{11}$$

or

$$p_t = (1 + \rho)^{-t}. \tag{11′}$$

Hence, for this problem, on the one hand, the dated prices of utility output are directly specified, by the hypothesis of a constant utility rate of interest, and the choice of initial utility as the numeraire commodity. On the other hand, the initial prices of capital stocks (and thereby saving–investment behavior) are only indirectly specified, by means of a so-called transversality condition,

$$\lim_{t\to\infty} q(t)\, k(t) = 0 \qquad \text{for} \quad \rho > 0, \tag{12}$$

or

$$\lim_{t\to\infty} q_t k_{t+1} = 0 \qquad \text{for} \quad \rho > 0. \tag{12′}$$

In other words, it is finally required that on an optimal path, the present value of the capital stocks tends to zero. Such a transversality condition has been shown to be necessary for optimality in the discrete-time case; see [21, 22, 37]. Under what circumstances it is also necessary for optimality in the continuous-time case is at present an open question. In either case it is clear that, in conjunction with perfect foresight, intertemporal profit maximization (given dated prices of utility output), the transversality condition amounts to a duality theorem for concave programming in some

appropriate infinite-dimensional space.[11] We plan in future research to investigate the necessity of the transversality condition in the continuous-time case by exploiting this duality characterization.

Formulation of the problem of optimal growth with zero discounting is much subtler, because for $\rho = 0$ no constrained maximum to (9) or (9′) will exist. If the maximand is changed to

$$\int_0^\infty [c(t) - c^*]\, dt, \tag{13}$$

or

$$\sum_{t=0}^{\infty} (c_t - c^*), \tag{13′}$$

where c^* is golden rule utility output, then a maximum may exist. As before, the laws of motion for the optimal path are given by (8) or (8′) and (11) or (11′). But now, the transversality condition, (12) or (12′), is no longer suitable. The condition appropriate to the maximand (13) or (13′) (again based on duality considerations) is instead

$$\lim_{t\to\infty} q(t)\, k(t) = Q^* k^*, \tag{14}$$

or

$$\lim_{t\to\infty} q_t k_{t+1} = Q^* k^*, \tag{14′}$$

[11] Namely, that maximizing (9) or (9′) subject to (2) or (2′) yields the same value as minimizing

$$\int_0^\infty w(t)\dot{l}(t)dt + q(0)\dot{k} \qquad \text{or} \qquad \sum_{t=0}^{\infty} w_t \dot{l}_t + q_{-1}\dot{k}$$

subject to (3) or (3′) plus the constraint (11) or (11′). That the two statements are equivalent is easily seen from analyzing the "feasible" behavior of

$$\int_0^T e^{-\rho t}\, c(t)dt + q(T)k(T) - \int_0^T w(t)\dot{l}(t)dt,$$

or

$$\sum_{t=0}^{T-1} (1+\rho)^{-t} c_t + q_{T-1}k_T - \sum_{t=0}^{T-1} w_t \dot{l}_t, \qquad \text{for} \quad T > 0.$$

Obviously, this particular duality theorem is only a special case of a general duality theorem involving arbitrary, exogenous specification of the weights attached to final output (e.g., terminal capital stocks as well as intermediate utility output).

In this connection, it should also be mentioned that our simplifying assumptions (A1) and (A3) play a role in this particular duality theorem similar to that of Slater's condition in the more familiar duality theorem for concave programming in Euclidean n-dimensional space, and can be weakened accordingly.

where k^* are the golden rule capital stocks, and Q^* the current investment goods prices (in terms of utility output) which support their maintenance.

As has been recognized for some time, the consumption-optimal growth with zero discounting and production-maximal growth models are very closely related, the differences being mainly in interpretation.[12] From our viewpoint, what is crucial is that both models yield canonical Hamiltonian dynamical systems, i.e., dynamical systems which are completely specified in terms of a time-autonomous Hamiltonian function. We have already seen that this is the case for consumption-optimal growth with zero discounting, since when $\rho = 0$, (11) or (11′) yields $p(t) = 1$ or $p_t = 1$, while by assumption $l(t) = 1$ or $l_t = 1$. For production-maximal growth such time-autonomy derives from the assumptions that there is no consumption goods output and that there is no primary factor constraint, which in our technological framework can be explicitly characterized by $c(t) = 0$ or $c_t = 0$, or better yet, what amounts to the same thing (because of free disposal in allocation (T8) or property (H3)),

$$p(t) = 0, \tag{15}$$

or

$$p_t = 0, \tag{15′}$$

and $l(t) = \infty$ or $l_t = \infty$. It is interesting to observe that (15) or (15′) already entails something like a transversality condition, since from (8) or (8′) we now have (here appealing to constant returns to scale (T5) or property (H3))

$$q(t)\, k(t) = q(0)\, \bar{k}, \tag{16}$$

or

$$q_t k_t = q_{-1} \bar{k}. \tag{16′}$$

And, indeed, it does turn out that, in effect, the production-maximal growth model is closed merely by postulating (15) or (15′), at least from the perspective of the type of stability analysis we will present (see our Sections 3C and 3D, below).

An alternative method of relating these two models is simply to convert consumption-optimal growth into production-maximal growth. This can be accomplished by treating both cumulated utility output and primary factor input (the latter conceived as a stock yielding service flows) as produced stocks, an approach nicely elaborated by several authors,

[12] We have in mind here the infinite horizon version of the production-maximal growth model, as analyzed, for instance, by Furuya and Inada [10]. Also, our assumptions about technology would have to be modified somewhat in order to be consistent with the discussion below. In particular, capital stock inputs must be limitational for the production-maximal growth model to make any sense.

notably Atsumi [1] and McKenzie [19]. Of course, the resulting model has some characteristics peculiar to itself, that is, that are not commonly found in the standard production-maximal growth models, and thus requires some analysis peculiar to itself.

We will not pursue either line of inquiry involving production-maximal growth models in the present paper, mainly because these models have been so extensively treated in the literature already, but also because the requisite modification of our global stability analysis for optimal growth models is fairly obvious.

Saving–investment constraints like (11) and (12), (11) and (14), and (15) or (11′) and (12′), (11′) and (14′), and (15′) do in fact close the Hamiltonian dynamical system, (8) or (8′), in the following sense: The set of initial prices of capital stocks for which there exists a path satisfying both the Hamiltonian dynamical system and the saving–investment constraints is severely restricted.[13] While this restriction is only indirect, it turns out, as we shall show in Section 3, sufficiently well-specified to render the stability analysis of the optimal (and maximal) growth models in terms of our Hamiltonian formalism almost trivial, basically because of the simple form taken by the utility output prices, (11) or (11′) (and (15) or (15′)).[14]

In descriptive growth theory the Hamiltonian dynamical system,[15] (8) or (8′), is typically closed by appending an instantaneous saving–investment relation of the general form

$$S\left(p(t), -\frac{\dot{p}(t)}{p(t)}, q(t), k(t), l(t)\right) = 0, \tag{17}$$

or

$$S\left(p_t, -\frac{p_t - p_{t-1}}{p_t}, q_t, k_t, l_t\right) = 0. \tag{17′}$$

[13] Regarding optimal growth, the severest restriction occurs when, for instance, T is endowed with sufficient additional structure so that M exhibits the following properties: (i) if $(1, q, r, w) \in M$, then $w > 0$, and (ii) if $(1, q^i, r^i, w^i) \in M$ for $i = 1, 2$ are distinct, then $c + [\lambda q^1 + (1 - \lambda)q^2]z - [\lambda r^1 + (1 - \lambda)r^2]k - [\lambda w^1 + (1 - \lambda)w^2] < 0$ for all $0 < \lambda < 1$, $(c, z, -k, -1) \in T$. Then the initial prices of capital stocks are unique. Less onerous additional structure will also suffice for this result.

[14] What renders the analysis so simple is not so much the fact that the interest rate, $-\dot{p}(t)/p(t)$ or $-(p_t - p_{t-1})/p_t$, is constant, as the fact that it is exogenous, that is, is determined independently of output prices and input stocks, $(p(t), q(t), k(t), l(t))$ or (p_t, q_t, k_t, l_t). Indeed, at the end of Section 3 we describe how our results are easily modified to cover the case where the interest rate is only asymptotically constant, though still exogenous.

[15] That competitive dynamical systems can be explicitly represented as Hamiltonian dynamical systems has been more or less commonly recognized among capital theorists for a number of years, especially in relation to prescriptive growth theory. Hahn's fairly specialized version of this fact [13] is the only published use of such representation in descriptive growth theory of which we are aware, though there may well be others.

For global stability analysis, the most critical feature of (17) or (17′) is that the interest rate, $-(\dot{p}(t)/p(t))$ or $-(p_t - p_{t-1})/p_t$, is endogenous, and, hence, varies with the evolution of the economy. A familiar example of such a saving–investment relation is the "Keynesian" hypothesis,

$$q(t)\, z(t) = s[r(t)\, k(t) + w(t)\, l(t)], \tag{18}$$

or

$$q_t z_t = s[r_t k_t + w_t l_t], \tag{18′}$$

where $0 < s < 1$ is the constant saving rate from NNI.[16] Another well-known example which can be cast in the form (17) or (17′) is the "Marxian" saving–investment hypothesis that wages are consumed, profits are reinvested,

$$q(t)\, z(t) = r(t)\, k(t), \tag{19}$$

or

$$q_t z_t = r_t k_t\,. \tag{19′}$$

In Section 4, we closely examine the stability properties of the descriptive growth model which incorporates the behavior described by (19) or (19′).[17]

Both (18) or (18′) and (19) or (19′) are particular variants of the general linear saving–investment hypothesis

$$q(t)\, z(t) = s_r r(t)\, k(t) + s_w w(t)\, l(t) - (1 - s_g)\, p(t)\, \frac{d[q(t)/p(t)]}{dt}\, k(t), \tag{20}$$

or

$$q_t z_t = s_r r_t k_t + s_w w_t l_t - (1 - s_g)\, p_t \left[\frac{q_t}{p_t} - \frac{q_{t-1}}{p_{t-1}}\right] k_t\,, \tag{20′}$$

where $0 \leqslant s_r, s_w, s_g \leqslant 1$ are the constant saving rates from alternative functional income streams, including capital gains relative to consumption,

$$p(t)\, \frac{d[q(t)/p(t)]}{dt}\, k(t) = \left[\dot{q}(t) - \frac{\dot{p}(t)}{p(t)}\, q(t)\right] k(t),$$

[16] Some versions of (18) or (18′) are in terms of gross rather than net quantities; see, for example, [32, 34] Of course, such a specification requires an explicit formulation of the way depreciation occurs, as well as of the technology for producing gross outputs. The standard formulation postulates constant depreciation rates (parallel with the linear form of the saving–investment hypothesis itself).

[17] Notice that Marxian saving–investment behavior has the special property that it can be *entirely* described in terms of the Hamiltonian function. Thus, stability for this model can *only* involve properties of the Hamiltonian function, a special circumstance, which, together with the particular form of Marxian saving–investment behavior, greatly simplifies its analysis.

or

$$p_t \left[\frac{q_t}{p_t} - \frac{q_{t-1}}{p_{t-1}}\right] k_t = \left[(q_t - q_{t-1}) - \frac{p_t - p_{t-1}}{p_t} \frac{p_t}{p_{t-1}} q_t\right] k_t .$$

Shell, Sidrauski, and Stiglitz [32] give an extended discussion of the rationale behind including capital gains in perceived income, and its implications for capital accumulation.

Clearly, there are many other interesting specifications of (17) or (17′). In any case, the general role of such a saving–investment relation, like that of the saving–investment constraints for the optimal and maximal growth models, is to restrict the permissible solutions to the general Hamiltonian dynamical system, hopefully, enough so that definite qualitative properties can be ascertained. For the purpose of analyzing stability, we have found it useful to conceive of (17) or (17′) as possibly restricting (8) or (8′) in several different ways: (i) by providing a direct additional restriction on the domain of the Hamiltonian function (and, hence, Hamiltonian dynamical system),

$$(p, q, k, l) \in D = \{(p, q, k, l): S(p, i, q, k, l) = 0\}; \tag{21}$$

(ii) by providing a direct or indirect endogenous determination of the interest rate, i (or, because without loss of generality $p(0) = 1$ or $p_0 = 1$, of the price of consumption goods output),

$$i \in I(p, q, k, l) \qquad \text{for} \quad (p, q, k, l) \in D; \tag{22}$$

and (iii) by providing an additional indirect restriction on the initial prices of capital stocks,

$$q(0) \in \bar{Q}(0) \qquad \text{or} \qquad q_{-1} \in \bar{Q}_{-1} . \tag{23}$$

As compared to the optimal or maximal growth models, stability of the descriptive growth model is potentially easier to analyze on account of (21), harder to analyze on account of (22). In our experience with a few leading cases, we have usually found the complexities stemming from (22) to outweigh the simplifications stemming from (21).

3. Stability of the Optimal Growth Model

A. *Preliminary Comments*

Our analysis of stability in the optimal growth model is conducted mainly in terms of continuous time; at the end of the section we briefly outline the parallel analysis in terms of discrete time. A substantial part of

the underlying results, those concerning existence and uniqueness of stationary points, is common to both. When doing so will cause no confusion, we will simply suppress the time variable.

For the optimal growth model, it turns out to be most convenient to conduct the analysis in terms of *current* rather than present values. We denote current prices and rentals by upper case letters,

$$(Q(t), R(t), W(t)) \equiv \frac{(q(t), r(t), w(t))}{p(t)}, \tag{24}$$

or

$$(Q_t, R_t, W_t) \equiv \frac{(q_t, r_t, w_t)}{p_t}. \tag{24'}$$

Then bringing together (8), (11), and (12) or (14), or (8′), (11′) and (12′), or (14′), and utilizing our other normalizing assumption (A1) along with this normalizing definition (24) or (24′), the competitive dynamical system for optimal growth can be written compactly as

$$\begin{cases} \dot{k} = \dfrac{\partial H(Q, k)}{\partial Q}, & k(0) = \bar{k}, \\ \dot{Q} = -\dfrac{\partial H(Q, k)}{\partial k} + \rho Q, & Q(0) \geqslant 0, \\ \lim\limits_{t\to\infty} Qe^{-\rho t}k = 0 & \text{for } \rho > 0, \\ \qquad\qquad = Q^*k^* & \text{for } \rho = 0, \end{cases} \tag{25}$$

or

$$\begin{cases} k_{t+1} = k_t + \dfrac{\partial H(Q_t, k_t)}{\partial H}, & k_0 = \bar{k}, \\ Q_{t+1} = Q_t - \dfrac{\partial H(Q_{t+1}, k_{t+1})}{\partial k} + \rho Q_t, & Q_{-1} \geqslant 0, \\ \lim\limits_{t\to\infty} Q_t(1+\rho)^{-t} k_{t+1} = 0 & \text{for } \rho > 0, \\ \qquad\qquad = Q^*k^* & \text{for } \rho = 0, \end{cases} \tag{25'}$$

where we define the current value Hamiltonian function $H(Q, k) \equiv H(1, Q, k, 1) \equiv H(1, q/p, k, 1)$, the current value of imputed NNP (= the current value of imputed NNI).

B. *Stationary Points*

As done earlier in our discussion of transversality conditions, we denote the variables corresponding to stationary points for the system (25) or (25′) with asterisks. Thus, in particular,

$$z^* = \frac{\partial H(Q^*, k^*)}{\partial Q} = 0,$$
$$-R^* + \rho Q^* = -\frac{\partial H(Q^*, k^*)}{\partial k} + \rho Q^* = 0. \tag{26}$$

Also, as is conventional, we will refer to stationary points, (Q^*, k^*), (and, occasionally and loosely, other of the variables corresponding to them, R^*, W^*, c^*, and z^*) as modified golden rule paths.

If the discount rate, ρ, is too high, then there may be no (nontrivial) modified golden rule path. However, utilizing our various assumptions about the technology it is fairly straightforward to establish the

EXISTENCE OF A MODIFIED GOLDEN RULE PATH. *Let*

$$\bar{\rho} = \sup\{\rho: (c, z, -k, -l) \in T, (c, z - \rho k) > 0\},$$

and suppose $\rho \in [0, \bar{\rho})$. *Then there exists* $(1, Q^*, R^*, W^*) \in M$ *and* $(c^*, z^*, -k^*, -1) \in T$ *such that* (i) $R^* = \rho Q^*$, (ii) $c^* > 0$, $z^* = 0$, *and* (iii) $c^* + Q^* z^* - R^* k^* - W^* = 0$.

Proof. The proof consists simply in carrying out the program outlined in [6, pp. 289–90]. In particular, we assert (without going into details) that (i) the mapping from the nonempty, compact, convex set $K = \{k: (c, z, -k, -l) \in T, \|(c, z, k)\| \leqslant B \text{ and } l \leqslant 1\}$ into itself defined by

$$\begin{aligned} \Phi(k) = \{k': (c', z', -k', -l') &\text{ is an optimal solution to} \\ \text{maximize } c' \quad &\text{subject to } (c', z', -k', -l') \in T, \|(c', z', k')\| \leqslant B, \\ z' - \rho k' \geqslant -\rho k, l' \leqslant 1\} \quad &\text{for } k \in K \end{aligned} \tag{27}$$

is a nonempty, convex-valued, upper semicontinuous correspondence (so that Kakutani's fixed point theorem can be appealed to), and (ii) a fixed point of the mapping (27), say k^*, yields a (nontrivial) modified golden rule path as optimal solutions to the dual concave and convex programming problems

$$\text{maximize } c \quad \text{subject to } (c, z, -k, -l) \in T, z - \rho k \geqslant -\rho k^*, l \leqslant 1,$$

and

$$\text{minimize } W + \rho Q k^* \quad \text{subject to } (p, Q, R, W) \in M, R - \rho Q = 0, p = 1.$$ [18]

[18] It is worth noting, referring to the property (B), that only the bound associated with replicable production points is actually relevant to this argument. And this particular bound enters in two ways, first, to provide the compactness required for the application of Kakutani's theorem, and second, to permit dropping the constraint $\|(c, z, k)\| < B$, once the existence of a fixed point has been established.

Since, just to be able to talk about stability, we will require the existence of some (nontrivial) modified golden rule path, hereafter we simply hypothesize:

EXISTENCE ASSUMPTION. $\rho \in [0, \bar{\rho})$. (E)

Notice that, as in the one-good model satisfying Inada's conditions, $\bar{\rho} = \infty$ is possible.

We next take up the question of the uniqueness of *the* modified golden rule capital stocks, k^*, which in our approach turns out closely related to the question of the convergence of the optimal paths defined by (25) to *some* modified golden rule capital stocks, $k^*, k^{**}, \ldots$. Our strategy is to find conditions in terms of the geometry of the current value Hamiltonian, $H(Q, k)$, which insure both uniqueness and convergence, the two properties which together define global stability, or simply, stability.

Though the zero discount rate case has been quite extensively treated in the literature, we begin our discussion with a full account of that special case. We do so partly to provide a self-contained, complete analysis of stability for optimal growth, but mostly to provide an intuitive, heuristic motivation for our later discussion of the general case. We should expressly warn the reader at the outset that, aside from the statement and verification of various uniqueness and stability conditions, the balance of this subsection is mostly devoted to motivating, explaining and interpreting, in other words, to exposition, not analysis.

If $\rho = 0$, then a stationary point, (Q^*, k^*) (now referred to as a golden rule path), must be a saddle point of the current value Hamiltonian, $H(Q, k)$,

$$H(Q^*, k) \leqslant H(Q^*, k^*) \leqslant H(Q, k^*) \qquad \text{for all} \quad (Q, k) \geqslant 0,$$

or

$$H(Q, k^*) - H(Q^*, k) \geqslant 0 \qquad \text{for all} \quad (Q, k) \geqslant 0, \tag{28}$$

since it satisfies (26), and $H(Q, k)$ is convex in Q and concave in k. Now suppose k^* were not unique, i.e., that (28) were also to hold with (Q^{**}, k^{**}) in place of (Q^*, k^*), where $k^{**} \neq k^*$. Then (28) would also achieve its minimum value of zero at (Q^{**}, k^{**}) as well as (Q^*, k^*). This suggests that any reasonably general condition which guarantees that k^* is unique must also, in effect, guarantee that (28) is a strict inequality for $k \neq k^*$.[19] Thus, one obvious sufficient condition for uniqueness is simply

[19] As Lionel McKenzie has emphasized, there are many specific technologies for which the last is not true, and yet k^* is unique. Uzawa's two-sector model is perhaps the best known of these (see Footnote 31 below). The point is that what we are concerned with here are conditions which don't require more detailed specification of the technology, for example, that it has an industry or sectoral structure.

that $H(Q, k)$ be strictly concave in k (symmetrically, that $H(Q, k)$ be strictly convex in Q is a sufficient condition for uniqueness of Q^*; see, for example, [25]). However, this is clearly stronger than needed, since the equally obvious direct condition, possibly valid without strict concavity, works just as well.

UNIQUENESS ASSUMPTION FOR $\rho = 0$.

$$k \neq k^* \Rightarrow H(Q, k^*) - H(Q^*, k) > 0, \tag{29}$$

or equivalently,[20]

$$k \neq k^* \Rightarrow (Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k} (k - k^*) > 0. \quad (\mathrm{U}^0)$$

(U^0) can be aptly called the *real Hamiltonian steepness condition*, since (29) means that looking from (Q^*, k^*), the Hamiltonian function, $H(Q, k)$, has negative steepness in all directions $(0, k - k^*) \neq 0$, or alternatively, that when $k \neq k^*$, the convex function $H(Q, k^*) - H(Q^*, k)$ lies strictly above its horizontal support at $(Q, k) = (Q^*, k^*)$.

Convergence to the golden rule capital stocks, k^*, can be established if our real Hamiltonian steepness condition holds uniformly in k.

STABILITY ASSUMPTION FOR $\rho = 0$. For every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\| k - k^* \| > \epsilon \Rightarrow (Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k} (k - k^*) > \delta. \quad (\mathrm{S}^0)$$

The proof of this assertion will be a particular application of the proof for the general model when $\rho \geqslant 0$, presented in the next subsection.

[20] This equivalence rests heavily on the convexity–concavity of $H(Q, k)$. On the one hand, (29) implies (U^0), since

$$\begin{aligned}(Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k} (k - k^*) &= \left[H(Q, k) + \frac{\partial H(Q, k)}{\partial k} (k^* - k)\right] \\ &\quad - \left[H(Q, k) + (Q^* - Q) \frac{\partial H(Q, k)}{\partial Q}\right] \\ &> H(Q, k^*) - H(Q^*, k).\end{aligned}$$

On the other hand, denial of (29) implies the denial of (U^0), since if $k' \neq k^*$ and $H(Q', k^*) - H(Q^*, k') = 0$, then $H(Q^*, k') = H(Q^*, k^*)$, $\partial H(Q^*, k')/\partial k = 0$, and $(Q^* - Q^*)(\partial H(Q^*, k')/\partial Q) - (\partial H(Q^*, k')/\partial k)(k - k^*) = 0$.

(U^0) and (S^0) can be interpreted in straightforward economic terms, and thereby related to the previous literature. Expanding the basic expression in both conditions yields

$$\begin{aligned}
&(Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) \\
&\quad = (Q - Q^*)\, z - R(k - k^*) \\
&\quad = (Q - Q^*)(z - z^*) - (R - R^*)(k - k^*) \\
&\qquad \text{(since by definition } z^* = R^* = 0) \\
&\quad = (1 - 1)(c - c^*) + (Q - Q^*)(z - z^*) \\
&\qquad - (R - R^*)(k - k^*) - (W - W^*)(1 - 1) \\
&\quad = -(c + Q^*z - R^*k - W^*) - (c^* + Qz^* - Rk^* - W) \\
&\qquad \text{(since by hypothesis} \\
&\qquad c + Qz - Rk - W = c^* + Q^*z^* - R^*k^* - W^* = 0).
\end{aligned} \tag{30}$$

Now let $x = (c, z, -k, -1)$ denote any efficient production point and $\pi = (1, Q, R, W)$ its associated current competitive prices. Then the last line in (30) becomes

$$-\pi x^* - \pi^* x, \tag{31}$$

the sum of the potential losses from adopting the golden rule production point x^* at current competitive prices π plus those from adopting the efficient production point x at golden rule prices π^*. Thus, the conditions (U^0) or (S^0) are nothing more than the requirements that these potential losses be positive or uniformly positive, respectively, when an efficient production point does not utilize golden rule capital stocks. In particular, (S^0) is therefore seen to be nothing but a symmetric adaptation of Radner's bounded value-loss condition (introduced in [23, pp. 101–2]) to the present model.[21]

We now turn to consideration of the general optimal growth model with constant, nonnegative discounting, $\rho \geqslant 0$. As we have just seen, the

[21] For the same sort of reason that (U^0) and (29) are equivalent, (S^0) is also easily shown to be equivalent to a direct analog of Radner's bounded value-loss condition, namely

$$\| k - k^* \| > \epsilon \Rightarrow \pi^* x > \delta.$$

Indeed, if we convert the consumption-optimal growth model with $\rho = 0$ into a production-maximal growth model in the manner suggested earlier, then the latter condition is precisely the appropriate bounded value-loss condition (that is, after taking account of how this artificial variant differs from the standard formulation of the production-maximal growth model).

analysis of uniqueness and convergence when $\rho = 0$ hinges on the fact that then (Q^*, k^*) is a saddlepoint of $H(Q, k)$. When $\rho > 0$, this saddlepoint property no longer obtains. Notice, however, that for the general model, a modified golden rule path, (Q^*, k^*), is *something like* a saddlepoint for the modified current value Hamiltonian function, $H(Q, k) - \rho Qk$. That is, from (26) and convexity–concavity, we have

$$H(Q^*, k) - \rho Q^* k \leqslant H(Q^*, k^*) - \rho Q^* k^* \leqslant H(Q, k^*) - \rho Q^* k^* \quad \text{for all} \quad (Q, k) \geqslant 0,$$

or

$$H(Q, k^*) - \rho Q^* k^* - [H(Q^*, k) - \rho Q^* k] \geqslant 0 \quad \text{for all} \quad (Q, k) \geqslant 0,$$

or

$$H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*) \geqslant 0 \quad \text{for all} \quad (Q, k) \geqslant 0. \tag{32}$$

Obviously, the reason we say "something like" has to do with the asymmetry of terms involving ρ. Applying reasoning similar to that leading to (29) and (U^0) now results in both strong and weak sufficient conditions for uniqueness of k^* (again because of asymmetry in terms involving ρ, which in effect makes the comparison involved in (U^s) below quantitative, rather than qualitative, as in that involved in (29) above.)

Uniqueness Assumption for $\rho \geqslant 0$ (Strong Version).

$$k \neq k^* \Rightarrow H(Q, k^*) - H(Q^*, k) + \rho Q(k - k^*) > 0,$$

or (U^s)

$$k \neq k^* \Rightarrow H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*)$$

and

Uniqueness Assumption for $\rho \geqslant 0$.

$$k \neq k^* \Rightarrow (Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*). \tag{U}$$

We repeat for emphasis: By convexity-concavity, (U^s) implies (U), but the converse implication is generally false unless $\rho = 0$, in which case (U) simply reduces to (U^0).

The geometric and economic interpretations of (U) are not nearly as transparent for $\rho > 0$ as they were for $\rho = 0$. Some idea of what (U)

means geometrically can be gleaned from the observation that its strengthening (U^s) amounts to a modified real Hamiltonian steepness condition: When $k \neq k^*$, the convex function $H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*)$ lies strictly above the indefinite quadratic $-\rho(Q - Q^*)(k - k^*)$. This interpretation also reemphasizes the self-evident importance of variation in Q (as well as k) in the statement (and verification) of either (U^s) or (U). Notice especially that both conditions are automatically satisfied whenever $-\rho(Q - Q^*)(k - k^*) < 0$ because of convexity–concavity and (32).

Again reasoning from the case $\rho = 0$, some idea of the economic content of (U) can be gotten from repeating the exercise which yielded (30), and which here leads to exactly the same expression, since on a modified golden rule path $z^* = 0$ but $R^* = \rho Q^*$. Thus, the fundamental inequality in (U) can be expressed as

$$-\pi x^* - \pi^* x > -\rho(Q - Q^*)(k - k^*)$$

or

$$-\pi x^* - \pi^* x > \rho Q^*(k - k^*) + \rho Q(k^* - k), \tag{33}$$

"the sum of potential losses exceeds the sum of potential interest." The idea embodied in (33) seems to revolve around what it denies; figuratively speaking, if (33) were not true, then there would appear to be potential gains in trade away from the modified golden rule path. Or putting it another way, and getting a little bit ahead of our story (see (S) and the subsequent argument below), society may not countenance growth toward the modified golden rule capital stocks if, when once achieved, it would then appear that society could actually have done better elsewhere.[22]

We close this subsection by again remarking that all the foregoing discussion applies equally well to both continuous time and discrete time.

C. *Stability Analysis*

We suppose that a solution to (25) exists. Then, consider the behavior of the (Lyapunov) current valuation function

$$V = (Q - Q^*)(k - k^*). \tag{34}$$

[22] A more concrete interpretation may be possible. When $\rho = 0$, (S^0) can be used to show that *any* path converging to the golden rule capital stocks is superior to *all* paths which do not (see, for example, [11]). When $\rho > 0$, no such very crude comparison can be coaxed from (S), since, so to speak, the initial evolution of a path weighs much more heavily than its terminal evolution. It is an interesting, open question whether, nevertheless, some much more delicate but just as easily interpretable comparison can be developed on the basis of (S).

Direct calculation shows that

$$\begin{aligned}
\frac{d(Ve^{-\rho t})}{dt} &\sim \dot{V} - \rho V \\
&= (Q - Q^*)\,\dot{k} + \dot{Q}(k - k^*) - \rho(Q - Q^*)(k - k^*) \\
&= (Q - Q^*)\,\frac{\partial H(Q, k)}{\partial Q} - \left(\frac{\partial H(Q, k)}{\partial k} - \rho Q\right)(k - k^*) \\
&\quad - \rho(Q - Q^*)(k - k^*) \\
&= (Q - Q^*)\,\frac{\partial H(Q, k)}{\partial Q} - \left(\frac{\partial H(Q, k)}{\partial k} - \rho Q^*\right)(k - k^*) \\
&\geqslant H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*) \\
&\geqslant 0, \qquad (35)
\end{aligned}$$

the last two inequalities again following from convexity–concavity of $H(Q, k)$ and the "saddlepoint" property of (Q^*, k^*), (32). Furthermore, the transversality condition in (25) requires that

$$\begin{aligned}
\lim_{t\to\infty} Ve^{-\rho t} &= \lim_{t\to\infty} (Q - Q^*)(k - k^*)\, e^{-\rho t} \\
&\leqslant \lim_{t\to\infty} (Qe^{-\rho t}k + Q^*e^{-\rho t}k^*) \\
&= 0 \qquad \text{for} \quad \rho > 0 \\
&(= 2Q^*k^* \qquad \text{for} \quad \rho = 0).
\end{aligned} \qquad (36)$$

Hence, since $Ve^{-\rho t}$ is increasing and nonpositive for $\rho > 0$ (bounded for $\rho = 0$), V must have an upper bound, say, $V \leqslant V^* < \infty$. If, in addition, we knew that $k \neq k^*$ entails $\dot{V} > 0$ (with some uniformity in the relation between the size of $(k - k^*)$ and the size of $\dot{V}$), then a routine argument (the details of which are provided below) would establish that

$$\lim_{t\to\infty} k = k^*.$$ [23]

[23] The reader may have begun to wonder why we focus on stability in terms of k rather than in terms of (Q, k). One reason is the fact that we don't even know for sure that Q is bounded (though $Qe^{-\rho t}$ likely is). Thus, the sort of analysis we present below concerning k cannot be carried out concerning Q. Perhaps a better (economic) reason is that we don't really care about the asymptotic behavior of prices in this context. The interesting aspect of the modified golden rule path is its real, not its price side, and we may well get convergence to the former even when Q^* is not unique. For example, the special model where $n = 1$ and $H(Q, k) = (1 + Q)f(k) - \lambda Qk$ can be employed to construct a solution to (25) on which, because k^* is, but Q^* is not unique, for given (Q^*, k^*), $\lim_{t\to\infty} k = k^*$, but $\lim_{t\to\infty} Q \neq Q^*$.

How do we insure the last? Casual inspection of the expressions in (35) suggests simply requiring that (U^s) or (U) hold uniformly in k.

STABILITY ASSUMPTION FOR $\rho \geqslant 0$ (STRONG VERSION). For every $\epsilon > 0$, there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow$

$$H(Q, k^*) - H(Q^*, k) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*) + \delta. \quad (S^s)$$

and

STABILITY ASSUMPTION FOR $\rho \geqslant 0$. For every $\epsilon > 0$, there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow$

$$(Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*) + \delta. \quad (S)$$

The reader may want to check for himself whether (S) holds in particular examples. Consider, for instance, the one-good model with a strictly concave intensive production function, $f(k) = F(k, 1)$, a constant depreciation rate, $f'(0) > \lambda > f'(\infty) \geqslant 0$, and utility output linear in consumption output, utility output $= c$ (without loss of generality). In this model $H(Q, k) = \max[1, Q]f(k) - \lambda Qk$, so that (S) holds when the discount rate, ρ, is sufficiently small. Notice that for this model, or its standard variants,[24] even our modified real Hamiltonian steepness condition (U) is far stronger than required for convergence. This is because, with a single capital good, the (Q, k) space is two-dimensional, while with both a single capital good and an intensive production technology which is strictly convex in capital intensity, the real side of an optimal path is unique. Thus, orbiting is not possible. Moreover, since the intensive production technology does not permit an unbounded capital intensity, the real side of an optimal path is also bounded. Hence, for such models, an optimal path must converge to *some* modified golden rule capital stock.

However, one should not be misled by these simple examples. In general, on optimal paths which are convergent, the valuation function, V, is also convergent, but not necessarily monotonically. This strongly suggests to us that any general argument for stability is going to require that V (or something very like it) be monotonically increasing, which in turn strongly

[24] See for example, [5, 15], or, for a pedagogic survey, [31]. A critical feature of all these models is that there is only a single capital good.

suggests that (S) (or something very like it) is the weakest general stability condition possible. Putting the matter another way, it seems very unlikely to us that, in more than two dimensions, orbiting on an optimal path can be ruled out unless some form of real Hamiltonian steepness obtains.[25]

Returning to the main point of this subsection, we now explicitly demonstrate the

STABILITY OF OPTIMAL GROWTH WITH CONSTANT, NONNEGATIVE DISCOUNTING. *Assumptions* (E) *and* (S) *imply that the real side of the solutions to* (25) *converges to unique* (*nontrivial*) *modified golden rule capital stocks.*

Proof. To begin with, we have from the definition of V and the stability condition (S) that

$$\begin{aligned}\dot{V} &= (Q - Q^*)\,\dot{k} + \dot{Q}(k - k^*)\\ &= (Q - Q^*)\,\frac{\partial H(Q, k)}{\partial k} - \left(\frac{\partial H(Q, k)}{\partial k} - \rho Q\right)(k - k^*)\\ &= \left[(Q - Q^*)\,\frac{\partial H(Q, k)}{\partial k} - \left(\frac{\partial H(Q, k)}{\partial k} + \rho Q^*\right)(k - k^*)\right]\\ &\quad + \rho(Q - Q^*)(k - k^*)\\ &\geqslant 0,\end{aligned}$$

and from (35) and (36) that

$$V \leqslant V^* < \infty.$$

Hence, putting these together, we have that

$$\lim_{t\to\infty} V = V^\infty \leqslant V^* < \infty.$$

Suppose that $\lim_{t\to\infty} k = k^*$ were not true, i.e., that for some $\epsilon > 0$, there were a sequence of points $\{t_j\}$ such that $\| k(t_j) - k^* \| > 2\epsilon$. Then, since (B) and (T7) entail uniform continuity of $k(t)$ on the half line $[0, \infty)$, there would also have to be a sequence of intervals $\{[\underline{t}_j, \bar{t}_j]\}$ such $\bar{t}_j - \underline{t}_j >$

[25] Perhaps this is an appropriate place to point out that, when stripped of our particular interpretation, our stability argument applies to any sufficiently regular dynamical system of the form (25), with or without convexity, provided the right-hand side of the inequality in (S) is further strengthened to read $\max[0, -\rho(Q - Q^*)(k - k^*) + \delta]$. (This strengthening incorporates the one direct implication of convexity–concavity actually used in the stability argument, namely, the final inequality in (35).) Of course, without convexity, neither the transversality condition (more generally, boundedness) nor the steepness condition (more generally, definiteness) typically makes much sense, which for us significantly reduces possible interest in such generalization.

$\Delta t > 0$ and $\| k(t) - k^* \| > \epsilon$ for $t \in [\underline{t}_j, \bar{t}_j]$. But this would imply, now utilizing condition (S) to bound uniformly the derivative

$$\dot{V}(t) = \Big[(Q(t) - Q^*) \frac{\partial H(Q(t), k(t))}{\partial Q} - \Big(\frac{\partial [H(Q(t), k(t))}{\partial k} + \rho Q^*\Big)(k(t) - k^*)\Big] + \rho(Q(t) - Q^*)(k(t) - k^*),$$

that there is a $\delta > 0$ such that

$$\dot{V}(t) > \delta \qquad \text{for} \quad t \in [\underline{t}_j, \bar{t}_j].$$

Hence, for sufficiently large t' we would have both $V^\infty - \delta \leqslant V(t) \leqslant V^\infty$ and

$$\begin{aligned} V(t) &\geqslant V(t') + \sum_{t' \leqslant \underline{t}_j < \bar{t}_j \leqslant t} \delta(\bar{t}_j - \underline{t}_j) \\ &\geqslant V(t') + [\max\{j\colon t' \leqslant \underline{t}_j \leqslant \bar{t}_j \leqslant t\} - \min\{j\colon t' \leqslant \underline{t}_j \leqslant \bar{t}_j \leqslant t\}]\, \delta \Delta t \end{aligned}$$

for $t \geqslant t'$, which are inconsistent, and the theorem is proved.

It should be clear that stronger "$\epsilon - \delta$" conditions would suffice for our argument, for instance (S^8), or the conditions used by Rockafellar [26], or the conditions used by Brock and Scheinkman [3].

We close this discussion by commenting briefly on the optimal growth model with a discount rate, $\rho(t) = -\dot{p}(t)/p(t)$, which is only asymptotically constant and nonnegative, $\lim_{t\to\infty} \rho(t) = \rho \geqslant 0$.[26] For this model, we have

$$\begin{aligned} \dot{V} &= (Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho(t)\, Q(k - k^*) \\ &= (Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho Q^*(k - k^*) \\ &\quad + \rho(Q - Q^*)(k - k^*) + (\rho(t) - \rho)(Q - Q^*)(k - k^*) \\ &\quad + (\rho(t) - \rho)\, Q^*(k - k^*). \end{aligned}$$

Since the last term in this expansion may be negative (the second to the last term may be negative too, but can eventually be safely neglected), our

[26] Symmetrically, we now assume that the valuation function, $V(t)$, is only uniformly bounded from above, $V(t) < V^* < \infty$ for all t. It is perhaps worth noting that this discussion applies equally well to any descriptive growth model for which these same properties, $\lim_{t\to\infty} \rho(t) = \rho > 0$ and $V(t) < V^* < \infty$ for all t, can be established.

argument above will not go through without some further strengthening of our modified real Hamiltonian steepness condition (U) beyond (S), basically to negate the effect of Q potentially being "too far" from Q^*. In particular, if we substitute the quantity $\max[\delta, -\delta(Q - Q^*)(k - k^*)]$ for the constant δ in the right-hand side of the inequality in (S), then the proof of stability involves only relatively minor modification, the details of which we omit.[27]

D. *Discrete Time*

Recall that the analysis of existence and uniqueness of stationary points is the same in either continuous time or discrete time. The reason for this fortunate circumstance is that, for an economy in balanced growth, currently compounding interest on this period's capital value (continuous time) is indistinguishable from accrued simple interest on last period's capital value (discrete time), provided both interest rates are the same. For the analysis of behavior away from stationary points, however, the intrinsic difference in the timing of interest calculations between continuous time and discrete time becomes of paramount importance.

In terms of modifying our analysis in the preceding subsection to accomodate discrete time, this shows up almost immediately once we write down the appropriate analogs to (34) and (35).

$$V_t = (Q_{t-1} - Q^*)(k_t - k^*) \tag{34'}$$

and

$$\begin{aligned}
&V_{t+1}(1+\rho)^{-(t+1)} - V_t(1+\rho)^{-t} \\
&\quad \sim V_{t+1} - (1+\rho)\, V_t \\
&\quad = (Q_t - Q^*)(k_{t+1} - k^*) - (1+\rho)(Q_{t-1} - Q^*)(k_t - k^*) \\
&\quad = (Q_t - Q^*)\left[\frac{\partial H(Q_t, k_t)}{\partial Q} + (k_t - k^*)\right] \\
&\qquad - (1+\rho)\left[\frac{1}{1+\rho}\left(\frac{\partial H(Q_t, k_t)}{\partial k} + Q_t\right) - Q^*\right](k_t - k^*) \\
&\quad = (Q_t - Q^*)\frac{\partial H(Q_t, k_t)}{\partial Q} - \frac{\partial H(Q_t, k_t)}{\partial k}(k_t - k^*) - \rho Q^*(k_t - k^*) \\
&\quad \geqslant H(Q_t, k^*) - H(Q^*, k_t) - \rho Q^*(k_t - k^*) \\
&\quad \geqslant 0.
\end{aligned} \tag{35'}$$

[27] Except to remark, for those interested in the details, that the essential difference comes in restricting attention to t sufficiently large so that

$$\max[|\rho(t) - \rho|, |\rho(t) - \rho| V^*, |(\rho(t) - \rho)Q^*(k - k^*)|] < \delta/3.$$

Equations (34′) and (35′) again lay a solid foundation for a routine argument establishing stability, provided only that the following inequality holds uniformly in $k_t \neq k^*$:

$$V_{t+1} - V_t = (Q_t - Q^*)\frac{\partial H(Q_t, k_t)}{\partial Q} - \frac{\partial H(Q_t, k_t)}{\partial k}(k_t - k^*) + \rho Q_{t+1}(k_t - k^*) > 0. \quad (37)$$

But notice the slippage of the time subscript in the last term of the middle expression! While this natural slippage highlights the fundamental character of the difference between continuous time and discrete time, the obvious method for getting around it also reemphasizes the central importance of both the price and quantity equations in our analysis. For, once more utilizing the second equation in (25′), we see that

$$\rho Q_{t+1}(k_t - k^*) = \frac{\rho}{1+\rho}\left(\frac{\partial H(Q_t, k_t)}{\partial Q} + Q_t\right)(k_t - k^*). \quad (38)$$

Hence, substituting from (38) into (37), and carrying out some algebraic rearrangement yields the direct analog of (S) for discrete time.

Stability Assumption for $\rho \geqslant 0$ (Discrete Time). For every $\epsilon > 0$, there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow$

$$(1+\rho)(Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) + \rho Q^*(k - k^*) > -\rho(Q - Q^*)(k - k^*) + \delta. \quad (S')$$

But now notice the addition of the factor $1 + \rho$ in the first term of the lefthand side!

Some brief reflection on the distinction between the stability conditions (S′) and (S) is revealing. Evidently, they will generally coincide only with a zero discount rate, $\rho = 0$. For a given current Hamiltonian function, $H(Q, k)$, and positive discount rate, $\rho > 0$, both, just one or the other, or neither may hold. Hence, in particular, (S′) may hold when (U), and, therefore, (S) as well, does not; in other words, for the system (25′), establishing convergence may be possible when establishing stability is not.[28] For example, in the one-good model mentioned in the last sub-

[28] Thus, it would be more accurate to refer to (S′) as a convergence, rather than a stability condition. However, we have yet to find a simple example in which (S′) holds, (U) does not, and there are actually several stationary capital stocks.

The reader should beware the fact that the comparison we are making here is somewhat artificial, since we are considering both continuous- and discrete-time conditions in terms of the *same* Hamiltonian function.

section, (S′) is in fact slightly weaker than (S). For another example, consider the (mathematically simplest) quadratic Hamiltonian model where $\nu = 1$ and $H(Q, k) = (\alpha/2)(Q - Q^*)^2 + \gamma(Q - Q^*)(k - k^*) - (\beta/2)(k - k^*)^2 + \rho Q^*(k - k^*)$ with $\alpha, \beta > 0$. Here (S′) amounts to the requirement that $4\alpha\beta > [(1 + \gamma)^2/(1 + \rho)]\,\rho^2$, but (U) or (S) to the requirement that $4\alpha\beta > \rho^2$. Clearly, since $-1 \leqslant \gamma \leqslant \infty$ entails $0 \leqslant (1 + \gamma)^2/(1 + \rho) \leqslant \infty$, any one of the four combinations of validity for these two parametric conditions is possible.

But then we really should not be surprised that the alternative treatments of time are just not the same. On the one hand, as we demonstrated earlier in the paper, the standard single capital good models must eventually converge to some modified golden rule capital stock in continuous time. On the other hand, Sutherland's illuminating example [35, example on p. 588] taught us some time ago that these same models may forever orbit in discrete time. (Incidentally, Sutherland's particular model essentially reduces to the quadratic Hamiltonian model with parameter values such that $4\alpha\beta = 23/16 < [(1 + \gamma)^2/(1 + \rho)]\,\rho^2 = 121/48 < \rho^2 = 4$.)

The general problem of relating results for continuous time with those for discrete time seems to us worth further, detailed investigation.

4. Stability of the Descriptive Growth Model with Marxian Saving–Investment Behavior

A. *Preliminary Comments*

It was our original hope that our Hamiltonian approach would provide a basis for unifying the stability analysis of maximal, optimal, and descriptive growth models. While we have not yet given up that hope, we are also not yet prepared to handle anything like the descriptive growth model with a general saving–investment relation (17) or (17′). Instead, the discussion of this section deals only with the Marxian saving–investment hypothesis, and that only in continuous time.

To the experienced capital theorist, it comes, of course, as no surprise that stability for descriptive growth models is much more fragile, that is, requires much more structure, than stability for optimal growth models. Even with a single capital good, perpetual orbiting in a descriptive growth setting is not an uncommon phenomenon. Just consider, for instance, the special saving–investment hypothesis (19) or (19′) we shall later concentrate attention on. Right away we have to contend with the well-known example of Inada [14], based on Uzawa's [36] famous two-sector model,

which shows that orbiting may arise whenever the investment-goods sector is more capital intensive than is the consumption-goods sector. Moreover, we must also contend with the fact that, in discrete time, even the one-good model with Marxian saving–investment behavior is likely to exhibit orbiting.[29]

In purely technical terms, there are major obstacles to simply extending the analysis of the previous section to descriptive growth models. For example, when the interest rate is endogenous and depends on the state of the economy, what particular value of the interest rate should form the basis for our stability condition (S)? We have seen that even the seemingly slight change from an exogenous, interest rate which is constant to one which is only asymptotically constant required some additional tightening of that condition. Or, for another example, what economic forces will insure boundedness of our valuation function in a decentralized, competitive dynamical system? There seems to be no intrinsic reason why every such system should mimic even a sufficiently constrained optimization problem, and therefore exhibit this more complete form of duality.

The worked-out examples of descriptive growth models all seem to have two distinctive characteristics: The underlying technology is simple enough so that it can be completely parameterized, while the saving–investment relation (typically it too can be completely parameterized) is special enough so that it severely restricts the domain of solutions to the Hamiltonian dynamical system. These special properties are then heavily exploited to show directly that paths which do not converge must eventually leave the permissible domain of solutions. Can these arguments even be systematically extended to a general model using our Hamiltonian approach? At the end of the section we will briefly comment on the extent of our success relative to the worked-out examples with Marxian saving–investment behavior (expecially [8, 14, 33, 36]), and on the possible future directions for applying our Hamiltonian approach to descriptive growth models with more general saving–investment behavior.

B. *The Formal Model*

In this section the choice of numeraire is fairly complicated. In general, the prices will be present values, but with the numeraire implicitly chosen so that the initial value of capital stocks equals the (constant) current value of capital stocks along the golden rule path. The express motivation

[29] In fact, given any three capital intensities, $0 < k_0 < k^* < k_1 < \infty$, it is easy to construct a strictly increasing, strictly concave intensive production function such that the golden rule capital stock is k^*, but the only solution for $k = k_0$ is

$$k_{2t} = k_0\,,\; k_{2t+1} = k_1 \qquad \text{for} \quad t = 0, 1, \ldots.$$

for this particular choice, like that for the subsequent analysis, is that our results generalize particular properties intrinsic to the worked-out examples mentioned above. However, at several points in the discussion (notably, in the statement of uniqueness and stability conditions) it will be convenient to revert to current values (recalling that (Q, R, W) $\equiv (q, r, w)/p$). Also, we will find it convenient to alternate freely between the Hamiltonian and conventional representations of the model. Finally, because we are only analyzing the model in continuous time, we will suppress the time variable.

The complete competitive dynamical system incorporating all of our maintained assumptions about technology, resources, and behavior can be written concisely as

$$\begin{cases} \dot{k} = \dfrac{\partial H(p, q, k, 1)}{\partial q} = \dfrac{\partial H(Q, k)}{\partial Q} = z, & k(0) = \bar{k} > 0, \\ \dot{q} = -\dfrac{\partial H(p, q, k, 1)}{\partial k} = -p\dfrac{\partial H(Q, k)}{\partial k} = -r, & q(0) \geqslant 0, \\ p\dfrac{\partial H(p, q, k, 1)}{\partial p} = pc = \dfrac{\partial H(p, q, k, 1)}{\partial l}\, 1 = w, \end{cases} \tag{39}$$

where again we use $H(Q, k) \equiv (H(1, q/p, k, 1)$. Since this model would not be a very interesting description of an economy otherwise, we append one further restriction on its solutions, that

$$w > 0, \tag{40}$$

the present wage rate, and, hence, current consumption output and its present value are all positive. The basis of our whole analysis is the simple observation that, because of linear homogeneity of H in (p, q) and the last equation in (39),

$$(\dot{qk}) = q\dot{k} + \dot{q}k = q\frac{\partial H(p, q, k, 1)}{\partial q} - \frac{\partial H(p, q, k, 1)}{\partial k}k = qz - rk =$$

$$(pc + qz - rk - w) - (pc - w) = -(pc - w) = 0$$

or

$$qk = q(0)\,\bar{k}, \text{ a constant.} \tag{41}$$

C. *Stationary Points*

As in our earlier discussion of optimal growth, we are interested in stationary points defined in terms of current values (even though our competitive

dynamical system is defined in terms of present values). The only stationary points of this sort which are consistent with (39)–(40) are those at which either the present value of investment-goods output is zero, i.e., the interest rate is not zero, or at which all present values are constant, i.e., the interest rate is zero. Existence of the latter, that is, of a golden rule path, is guaranteed by our earlier argument. Thus, for our purposes now, we need a somewhat different uniqueness assumption, namely the

POSITIVITY AND UNIQUENESS ASSUMPTION. (i) $Q^*k^* > 0$, and (ii)

$$k \neq k^* \Rightarrow (Q - Q^*)\frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k}(k - k^*) = (Q - Q^*)\, z - R(k - k^*) > 0 \quad \text{(P)}$$

when $Q(\partial H(Q, k)/\partial Q) = Qz = (\partial H(Q, k)/\partial k)\, k = Rk$.

Various additional restrictions on the technology (having to do with the necessity and productivity of capital) would entail positivity $Q^*k^* > 0$. The second part of (P) is essentially a weakening of (U^0) which focuses on just those perturbations satisfying the saving–investment hypothesis; it would obviously be satisfied whenever (U^0) is.

Given the first part of (P), if $q(0)\,\bar{k} > 0$, then the price side of the solutions to (39) can be normalized so that $qk = q(0)\,\bar{k} = Q^*k^*$.[30] Thus, we have either

$$qk = 0 \qquad \text{or} \qquad qk = Q^*k^* > 0. \tag{42}$$

D. *Stability Analysis*

Here, we focus our attention on a current valuation function, $v = (q - Q^*)(k - k^*)$. For this system, from (42), v is bounded by

$$v = (q - Q^*)(k - k^*) = qk - qk^* - Q^*k + Q^*k^* \leqslant 2Q^*k^* < \infty, \tag{43}$$

while, from (39) and (41), v behaves according to

$$\begin{aligned} \dot{v} &= \dot{q}(k - k^*) + (q - Q^*)\,\dot{k} \\ &= (\dot{qk}) - \dot{q}k^* - Q^*\dot{k} \\ &= \frac{\partial H(p, q, k, 1)}{\partial k}\,k^* - Q^*\,\frac{\partial H(p, q, k, 1)}{\partial q} \\ &= rk^* - Q^*z. \end{aligned} \tag{44}$$

[30] Otherwise any normalization, e.g., $p(0) = 1$, will do.

In order for $\dot{v}$ to be sufficiently positive to guarantee convergence of k to k^*, we also need a somewhat different stability assumption than before, namely, the

INTENSITY AND STABILITY ASSUMPTION. (i) $(p, q, r, w) \in M$, $(c, z, -k, -l) \in T$, $pc + qz - rk - w = 0$, $pc = w > 0$, and $qk = Q^*k^* > 0 \Rightarrow rk^* \{\gtreqless\} 0$ according as $p \{\gtreqless\} 1$, or $rk^* \geqslant Rk^*$, and (ii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow$

$$(Q - Q^*) \frac{\partial H(Q, k)}{\partial Q} - \frac{\partial H(Q, k)}{\partial k} (k - k^*) = (Q - Q^*) z - R(k - k^*) > \delta \quad \text{(I)}$$

when $Q(\partial H(Q, k)/\partial Q) = Qz = (\partial H(Q, k)/\partial k)\, k = Rk$.

We should emphasize that it is the first part of (I) which is really fundamental, since it entails that (44) can be bounded below by

$$rk^* - Q^*z \geqslant Rk^* - Q^*z, \tag{45}$$

and in that condition, the normalization $qk = Q^*k^*$ which is really crucial, since, without it, finding fairly general stability conditions seems an almost impossible task. Again, this particular normalization was suggested by careful scrutiny of the worked-out examples, especially the one- and two-sector models. Notice, by way of interpretation, that the conclusion of the first part of (I) can be restated as (because $Qk \{\lesseqgtr\} Q^*k^*$ according as $p \{\gtreqless\} 1$ when $qk = Q^*k^*$)

$$Rk^* \{\gtreqless\} 0 \qquad \text{according as} \quad Qk \{\lesseqgtr\} Q^*k^*, \tag{46}$$

"aggregate" net rentals are positive or negative depending on whether "aggregate" capital intensity is less or greater than its golden rule path value.

A reader steeped in the tradition of neoclassical growth theory might plausibly conjecture that (46) is closely related to the condition that

$$c \{\lesseqgtr\} c^* \qquad \text{according as} \quad Qk \{\lesseqgtr\} Q^*k^*, \tag{47}$$

aggregate consumption is less or greater than its golden rule path value depending on whether "aggregate" capital intensity is. Unfortunately, neither (46) nor (47) implies the other. It is interesting to notice, however, that (47) can be strengthened to provide an alternative stability condition,

since (44) can also be bounded below by

$$\begin{aligned} rk^* - Q^*z &= -(pc^* + qz^* - rk^* - w) - (c + Q^*z - R^*k - W^*) \\ &\qquad - (p-1)(c - c^*) \\ &\geqslant -(p-1)(c - c^*). \end{aligned} \tag{48}$$

By virtue of (48), (47) entails that (again because $Qk \{\gtreqless\} Q^*k^*$ according as $p \{\gtreqless\} 1$ when $qk = Q^*k^*$)

$$rk^* - Q^*z \{\geqq 0\} \qquad \text{according as} \quad Qk \{\overset{\neq}{=}\} Q^*k^*, \tag{49}$$

a condition which can obviously be elaborated to yield $\dot{v}$ sufficiently positive to guarantee convergence of k to k^*. In either case, (46) or (47), the parallel with intrinsic properties of the one-good model is self-evident.

We are now in a position to demonstrate the

STABILITY OF DESCRIPTIVE GROWTH WITH MARXIAN SAVING–INVESTMENT BEHAVIOR. *Assumptions* (P) *and* (I) *imply that the real side of the solutions to* (39) *converges to unique* (*nontrivial*) *golden rule capital stocks.*

Proof. According to (42), there are two distinct cases to consider. (i) $qk = 0$: Here $k > 0$ implies $q = 0$ implies $r = 0$ (as $\dot{q} = -r$) implies $rk^* = 0$. Hence, from (44) $\dot{v} = -Q^*z$, while from just the second part of (I)

for every $\epsilon > 0$ there is a $\delta > 0$ such that $\| k - k^* \| > \epsilon \Rightarrow -Q^*z > \delta$.

Thus, by the sort of reasoning used in our earlier stability argument, (43) entails $\lim_{t\to\infty} k = k^*$. (ii) $qk = Q^*k^* > 0$: Here, for any point along a solution to (39)–(40), the first part of (I) allows us to strengthen the second part to

$$\text{for every } \epsilon > 0 \text{ there is a } \delta > 0 \text{ such that } \| k - k^* \| > \epsilon \Rightarrow \dot{v} = rk^* - Q^*z \geqslant Rk^* - Q^*z > \delta.$$

Thus, as before, (43) entails $\lim_{t\to\infty} k = k^*$.

E. *Worked-out Examples and Extensions*

It is fairly easy to show that (P) and (I) are satisfied in Uzawa's two-sector model, provided that the consumption-goods sector is more capital intensive than the investment-goods sector so that Inada's type of counter-example is excluded.[31] It is much less obvious whether our conditions

[31] We note in passing that this model does not satisfy a stronger uniqueness or stability assumption, like (U^0) or (S^0) of the last section. Many feasible allocations with $k \neq k^*$ can be static profit maximizing at (modified) golden rule prices.

apply to the other worked-out examples. Thus far we have been able to show, by an extremely tedious argument, that the Shell–Stiglitz example [33] satisfies (P) and a weaker version of (I), which may imply stability as well, but at present it appears that it does not satisfy (I). We have not yet tackled the other worked-out examples (in particular, the Caton–Shell example [8]). Even for the descriptive growth model with Marxian saving–investment behavior, much more work needs to be done!

Analysis of this model is clearly simplified because the saving–investment hypothesis can be described entirely in terms of H, and because the only consistent, meaningful stationary points are golden rule paths. The next obvious candidate for analysis is a model with the Keynesian saving–investment hypothesis, which has neither of these special features. Our failure thus far to find general results for such a model seems to be related to our inability to see how to utilize the restrictions imposed by (18), if indeed they can be. In any case, the investigation of this, and further similar extensions, seem a fascinating and important task for future research.

Of course, it may well turn out that the stability notion used in our present analysis is simply far too rigid for any such contemplated extensions. For example, it may be more appropriate to ask whether there is some bounded set of capital stocks, perhaps exhibiting other special properties, which serves as a global attractor for the real side of the solutions to a Hamiltonian dynamical system. This broadened stability notion would then obviously accommodate such phenomena as arise in Sutherland's or Inada's examples. It might also be sufficient to completely dispel Hahn-like worries [12] about the "gross" inefficiency of capitalist development. However, on the other hand, it would definitely entail a significant loss in our ability to do meaningful comparative dynamics.

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REFERENCES

1. H. ATSUMI, Neoclassical growth and the efficient program of capital accumulation, *Rev. Econ. Stud.* **32** (1965), 127–136.
2. W. A. BROCK, On the existence of weakly maximal programmes in a multi-sector economy, *Rev. Econ. Stud.* **37** (1970), 275–280.
3. W. A. BROCK AND J. A. SCHEINKMAN, Global asymptotic stability of optimal control systems with applications to the theory of economic growth, *J. Econ. Theory* Symposium on "Hamiltonian Dynamics in Economics," **12** (1976), 164–190.
4. E. BURMEISTER, C. CATON, A. R. DOBELL, AND S. ROSS, The "saddlepoint property" and the structure of dynamic heterogeneous capital good models, *Econometrica* **41** (1973), 79–95.
5. D. CASS, Optimum growth in an aggregative model of capital accumulation, *Rev. Econ. Stud.* **32** (1965), 233–240.
6. D. CASS, Duality: a symmetric approach from the economist's vantage point, *J. Econ. Theory* **7** (1974), 272–295.
7. D. CASS, The Hamiltonian representation of static competitive or efficient allocation, *in* "Essays in Modern Capital Theory" (M. Brown, K. Sato and P. Zarembka, Eds.), pp. 159–178, North–Holland, Amsterdam, 1976.
8. C. CATON AND K. SHELL, An exercise in the theory of heterogeneous capital accumulation, *Rev. Econ. Stud.* **38** (1971), 13–22.
9. R. DORFMAN, P. A. SAMUELSON, AND R. M. SOLOW, "Linear Programming and Economic Analysis," McGraw–Hill, New York, 1958.
10. H. FURUYA AND K. INADA, Balanced growth and intertemporal efficiency in capital accumulation, *Int. Econ. Rev.* **3** (1962), 94–107.
11. D. GALE, On optimal development in a multi-sector economy, *Rev. Econ. Stud.* **34** (1967), 1–18.
12. F. H. HAHN, Equilibrium dynamics with heterogeneous capital goods, *Quart. J. Econ.* **80** (1966), 633–646.
13. F. H. HAHN, On some equilibrium paths, *in* "Models of Economic Growth" (J. A. Mirrlees and N. H. Stern, Eds.), pp. 193–206, Macmillan, London, 1973.
14. K. INADA, On a two-sector model of economic growth: comments and a generalization, *Rev. Econ. Stud.* **30** (1963), 119–127.
15. M. KURZ, Optimal economic growth and wealth effects, *Int. Econ. Rev.* **9** (1968), 155–174.
16. L. LAU, A characterization of the normalized restricted profit function, *J. Econ. Theory* Symposium on "Hamiltonian Dynamics in Economics," **12** (1976), 131–163.
17. E. MALINVAUD, Capital accumulation and efficient allocation of resources, *Econometrica* **21** (1953), 233–268.
18. L. W. MCKENZIE, Turnpike theorems for a generalized Leontief model, *Econometrica* **31** (1963), 165–180.
19. L. W. MCKENZIE, Accumulation programs of maximum utility and the von Neumann facet, *in* "Value, Capital and Growth" (J. Wolfe, Ed.), pp. 353–383, Edinburgh Univ. Press, Edinburgh, 1968.
20. M. MORISHIMA, Proof of a turnpike theorem: the "no joint production case," *Rev. Econ. Stud.* **28** (1961), 89–97.
21. B. PELEG, Efficiency prices for optimal consumption plans, *J. Math. Anal. Appl.* **29** (1970), 83–90.
22. B. PELEG AND H. E. RYDER, JR., On optimal consumption plans in a multi-sector economy, *Rev. Econ. Stud.* **39** (1972), 159–169.

23. R. RADNER, Paths of economic growth that are optimal with regard only to final states: a turnpike theorem, *Rev. Econ. Stud.* **28** (1961), 98–104.
24. F. P. RAMSEY, A mathematical theory of saving, *Econ. J.* **38** (1928), 543–559.
25. R. T. ROCKAFELLAR, Saddlepoints of Hamiltonian systems in convex problems of Lagrange, *J. Optimization Theory Appl.* **12** (1973), 367–390.
26. R. T. ROCKAFELLAR, Saddlepoints of Hamiltonian systems in convex Lagrange problems having a nonzero discount rate, *J. Econ. Theory* Symposium on "Hamiltonian Dynamics in Economics," **12** (1976), 71–113.
27. H. E. RYDER, JR., Optimal accumulation in a two-sector neoclassical economy with non-shiftable capital, *J. Polit. Econ.* **77** (1969), 665–683.
28. H. E. RYDER, JR., AND G. M. HEAL, Optimum growth with intertemporally dependent preferences, *Rev. Econ. Stud.* **40** (1973), 1–31.
29. P. A. SAMUELSON, The general saddlepoint property of optimal-control motions, *J. Econ. Theory* **5** (1972), 102–120.
30. J. A. SCHEINKMAN, On optimal steady states of *n*-sector growth models when utility is discounted, *J. Econ. Theory* Symposium on "Hamiltonian Dynamics in Economics," **12** (1976), 11–30.
31. K. SHELL, Applications of Pontryagin's maximum principle to economics, *in* "Mathematical Systems Theory and Economics I" (H. W. Kuhn and G. P. Szegö, Eds.), pp. 241–292, Springer–Verlag, Berlin, 1969.
32. K. SHELL, M. SIDRAUSKI, AND J. E. STIGLITZ, Capital gains, income, and saving, *Rev. Econ. Stud.* **36** (1969), 15–26.
33. K. SHELL AND J. E. STIGLITZ, The allocation of investment in a dynamic economy, *Quart. J. Econ.* **81** (1967), 592–609.
34. R. M. SOLOW, A contribution to the theory of economic growth, *Quart. J. Econ.* **70** (1956), 64–95.
35. W. R. S. SUTHERLAND, On optimal development in a multi-sectoral economy: the discounted case, *Rev. Econ. Stud.* **37** (1970), 585–589.
36. H. UZAWA, On a two-sector model of economic growth, *Rev. Econ. Stud.* **28** (1961), 40–47.
37. M. L. WEITZMAN, Duality theory for infinite horizon convex models, *Management Sci.* **19** (1973), 783–789.