ESSAY *I*

Taste and Quality Change in the Pure Theory of the True Cost-of-Living Index

1. *Introduction*

The standard theory of the true cost-of-living index gives a rather uncomfortable treatment to taste and quality changes (including the introduction of new goods). The consumer is assumed to have always had an unchanging indifference map, complete with axes for all new goods of whose potential existence he in fact was not aware before their introduction. Similarly, quality change is treated either as an introduction of a new good or as a simple repackaging of an old one equivalent to a price reduction.[1] Yet the justification for the latter procedure has never been satisfactorily set forth, while the former one meets with many of the same difficulties as does the treatment of new goods itself.

If the treatment of new goods and quality change is less than fully satisfactory, however, the treatment of taste change is nonexistent. The assumption of an unchanging indifference map even defined over nonexistent goods is apparently crucial for a theory which is often erroneously thought to answer the question: How much would it cost in today's prices to make the consumer just as well off as he was yesterday? This question cannot be answered without resorting to an arbitrary intertemporal weighting of utilities. Yet taste changes do occur and the cost-of-living index is often carelessly thought to be designed to answer that question (Nat. Bur. Econ. Res., 1961, pp. 51–9; v. Hofsten, 1952).

This essay begins by arguing that the difficulty is due only to a misinterpretation of the theory of the true cost-of-living index. That theory does not in fact seek to answer the question posed above, nor does it make intertemporal comparisons of utility. Indeed, we observe that such a question can never be answered and such comparisons never made because they have no operational content. Incautious application of the theory has avoided facing up to this by the use of an apparently appealing but completely arbitrary and untestable hidden assumption which does no apparent harm when tastes are constant but which breaks down utterly when tastes do change.

That assumption, however, is not part of the theory and the question which the theory does answer retains its meaning whether or not tastes are constant. The pure theory of the cost-of-living index, rigorously interpreted, accommodates taste changes quite comfortably.

Accordingly, we then go on to consider a case of parametrizable taste change in full detail. That case can be given the interpretation of consumers learning more about the properties of a recently introduced good. We derive the consequences for index number construction of such a circumstance.

Moreover, the rigorous formulation of the theory involved in the treatment of taste change aids also in the treatment of new goods and of quality change. It does so in two ways. First, the formally acceptable but practically uncomfortable assumption that the consumer has always known about unavailable goods and qualities disappears. Second, by focusing attention on a proper question, the analysis of new goods and quality change becomes relatively straightforward. While it is true in principle that (unlike the case of taste change) the same analysis could be carried out without so rigorous a formulation (given the assumption of unchanging tastes for nonexistent goods), that formulation makes it very clear what is involved. Asking the right question is a good part of obtaining the answer.

Thus the last two sections of the essay discuss the treatment of new goods and of quality change respectively and show what kind of information is needed for the handling of these problems in a satisfactory manner.

II. *The Theory of the True Cost-of-Living Index and Intertemporal Comparisons of Welfare*

As indicated, a frequently encountered view of the true cost-of-living index is that it is designed to answer the question: 'What income would be required to make a consumer faced with today's prices just as well off as he was yesterday when he faced yesterday's income and yesterday's prices?' The difficulty that is presented by taste changes in answering this question is immediately apparent. What is meant by 'just as well off as he was yesterday' if the indifference map has shifted?

Yet reflection on this issue shows that the same difficulty appears even if tastes do not change. While it is apparently natural to say that a man whose tastes have remained constant is just as well off today as he was yesterday if he is on the same indifference curve in both periods, the appeal of that proposition is no more than apparent. In both periods, the man's utility function is determined only up to a monotonic transformation; how can we possibly know whether the level of true utility (whatever that may mean) corresponding to a given indifference curve is the same in both periods? The man's efficiency as a pleasure-machine may have changed without changing his tastes.

Indeed, we have no more justification for saying that a man on the same indifference curve at two different times is equally well off at both than we do for saying that two men who happen to have the same indifference map are equally well off if they have the same possessions. Both statements are attractive for reasons of simplicity and both are completely without any operational content whatsoever. One never steps into the same river twice and the comparison between a man's utility now and his utility yesterday stands on precisely the same lack of footing as the comparison of the utilities of two different men.

Thus, a consideration of the problem of taste change on this interpretation of the theory of the true cost-of-living index merely makes explicit a problem that is apparently there all the time. If that theory were really founded upon intertemporal comparisons of utility of the type described, then that theory would be without foundation.[2]

In fact, however, the theory of the true cost-of-living index makes no such comparisons, and rigorous statements of that theory have avoided them. Such statements run as follows: 'Given an indifference map, we compare two *hypothetical* situations, A and B. We ask how much income the consumer in B would require to make him just indifferent between facing B's prices and facing A's prices with a stated income.' Note that the question of whether the consumer has the same utility in A as in B never arises. So long as we remain on this level of abstraction, the point in time and space at which the consumer has the indifference map used in the comparison may be A or it may be B or it may be any other *single* point different from both of these.

In practice, however, the cost-of-living index is meant to compare two real situations rather than two hypothetical ones and A and B become, for example, yesterday and today, respectively. In this case, it is natural to take the indifference map to be used as the one in force at either A or B,[3] and if tastes have not changed so that the two maps are the same, it is easy to slip into the erroneous (but in this case harmless) usage of saying that what is compared are the relative costs of making the consumer at B just as well off as he was at A. If the indifference maps differ, however, such a slip is dangerous and it must never be forgotten that the viewpoint from which the comparison is made is not necessarily identical with either A or B.

Thus, the true cost-of-living index is supposed to represent a comparison between two opportunity or constraint loci not between two utility levels. The first constraint locus is that given by yesterday's income and prices – it is yesterday's budget constraint. The second is a budget constraint defined by today's prices but with income a parameter. The true cost-of-living index does not answer the question: 'How much income would it take today to make me as well off as I was yesterday with yesterday's budget constraint?' That question is unanswerable. A similar-seeming question which *can* be answered is: 'How much income is required *today* to make me just indifferent between facing yesterday's budget constraint and facing a budget constraint defined by today's prices and the income in question?' The latter question refers to a choice which can in principle be posed; the former does not.

Note further that the question just posed retains its meaning even if tastes have changed between yesterday and today. It is a question posed entirely in terms of today's tastes and involves a comparison of present and past *constraints*, not a comparison of present and past utilities. As it were, we replace the question: 'Were you happier when young than you are now?' with the question: 'Would you like to live your youth over again, having the tastes you do now?' The latter question may seem more fanciful than the former, but it is the one which is operationally meaningful.

It is evident, however, that a second question can also be posed, the answer to which may differ from that to the question just suggested if tastes in fact change. That question is: 'What income would have been required *yesterday* to make you just indifferent between facing yesterday's budget constraint and facing a budget constraint defined by today's prices and the income in question?' This is the same question as before from the vantage of yesterday's tastes rather than today's. It is equally meaningful, but, we shall argue below, not as interesting.[4]

If tastes do not change, then the answers to the two questions coincide.

In this case also, it is obvious that the required income is precisely that income which would place the consumer today on the same indifference curve as he achieved yesterday. Thus, in the case of no taste change, the cost-of-living index implied by the answers to our questions is precisely that given by the erroneous application of the traditional theory. As indicated in the introduction, however, even in the case of no taste change the advantage of a rigorous formulation is more than aesthetic, since, by focusing attention on a choice between alternative constraints, such a formulation aids in the treatment of problems such as the incorporation of new goods or quality change into the cost-of-living index.

What about the case of taste change, however, in which we have asked two parallel but different questions which (in this case) have two different answers in general? It seems clear that when intertemporal problems are involved, the asymmetry of time makes the question asked assuming today's tastes more relevant than the equally meaningful question asked assuming yesterday's tastes.[5] That this is so may be seen from the following example.

Consider two alternative time paths of prices with the same initial values. In the first, path A, the cost-of-living index considered from the point of view of yesterday's tastes rises, while that considered from the point of view of current tastes stays constant or falls; in the second, path B, the reverse is true. It is clear that the consumer will be better off in every period under path A than under path B, or, equivalently, that in every period, the cost-of-living is higher on path B than on path A. Faced with a choice, rational policy should prefer path A to path B.[6] Indeed, every practical question which one wants the cost-of-living index to answer is answered with reference to current, not base-year tastes. Succinctly, if the prices of goods no longer desired rise and those of goods newly desired fall, a cost-of-living index should fall, not rise. The question of how a man with base-year tastes would view the matter is an operationally meaningful one; it is not a terribly relevant one, however, save insofar as it casts light on the cost of living viewed with current tastes.

This argument has an immediate corollary. The general practice in the construction of consumer price indices is to use Laspeyres indices with base-period quantity weights rather than Paasche indices with current weights. In the case of no taste change, a frequently encountered proposition is that a Laspeyres index overstates price rises and a Paasche index understates them, because of the inadequate treatment afforded substitution effects.[7] If tastes change, however, and if we agree that it is the current-taste cost-of-living in which we are interested, a Laspeyres index loses much of its meaning. That index is a relevant upper bound for a true cost-of-living index with base-year tastes; it need not be such a bound for a true cost-of-living index with current tastes. A Paasche index, on the other hand, retains its property of being a lower bound on the current-tastes index (but may lose it for the base-year-taste index). When tastes change, Laspeyres and Paasche indices cease to become approximations to the same thing and become approximations to different things. As we have just seen, it is the Paasche index which approximates the relevant magnitude; the Laspeyres index becomes less relevant.

Indeed, such relevance as is retained by a Laspeyres index occurs only if taste changes take place in such a way as to make a base-year-taste index differ from a current-taste index in some specific way. If one is willing to specify *how* tastes change and to parametrize that specification, one may obtain results on how a Laspeyres index should be adjusted. This is done for a specific class of cases in the next section. If one is not willing to make such a specification, but believes that important taste changes have taken place, one should put more reliance on a Paasche index and less on a Laspeyres than has traditionally been done.[8]

Before closing this section, it may be well to formalize the question which, we have argued, the true cost-of-living index is designed to answer. Given base-period prices of goods $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n$, base-period income $\hat{y}$, current prices of goods $p_1, p_2, \ldots, p_n$, the problem is to find that income y such that the representative consumer is *currently* indifferent between facing current prices with income y and facing base-period prices with base-period income. The true cost-of-living index is then $(y/\hat{y})$.

Let $u(\cdot)$ be an ordinal utility function derived from the representative consumer's current preference map. The problem reduces to solving for the nonnegative values of $x_1, x_2, \ldots, x_n$, that minimize the expression

$$y = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n , \tag{2.1}$$

where x_i ($i = 1,2,\ldots,n$) is the amount of the ith good that would be purchased at current prices and income y, subject to the requirement that

$$u(x_1, x_2, \ldots, x_n) = u(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) . \tag{2.2}$$

$\hat{x}_i$ $(i = 1,2,\ldots,n)$ is the amount of the ith good that currently would be purchased if the consumer faced base-period prices with base-period income. That is, nonnegative $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ are chosen to maximize utility

$$u(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n) \tag{2.3}$$

subject to the budget constraint

$$\hat{y} \geqslant \hat{p}_1\hat{x}_1 + \hat{p}_2\hat{x}_2 + \ldots + \hat{p}_n\hat{x}_n\,. \tag{2.4}$$

It may be noted that a more compact formulation can be given in terms of the indirect utility function (Houthakker, 1951–2, 157–63). Thus, let $\phi(p_1,p_2,\ldots,p_n,y)$ be the indirect utility function, so that $\phi(p_1,p_2,\ldots,p_n,y)$ is the maximal value of $u(x_1,x_2,\ldots,x_n)$ subject to $\sum_1^n p_i x_i = y$. The cost-of-living index is $(y/\hat{y})$, where y is the solution to $\phi(p_1,p_2,\ldots,p_n,y) = \phi(\hat{p}_1,\hat{p}_2,\ldots,\hat{p}_n,\hat{y})$. We have not used this formulation in what follows since taste changes seem to be parametrizable more easily in terms of the direct than in terms of the indirect utility function and because we shall later work with more complicated constraints. However, the properties of the indirect utility function may be useful for future work in this area.

III. *Taste Change*

Consumers' tastes change for a variety of reasons some of which are so mysterious to the ordinary economist that he is unlikely to offer much in the way of a systematic understanding. But certain instances of taste change possess a more systematic structure. For example, it may be known to be the case that a recently introduced electrical appliance, say, increases monotonically in desirability through time during the period in which consumers are learning about the usefulness of the appliance. In such a case, one unit of the appliance in a later year may afford the same service as more than one in an earlier year because of the increase in consumer information but with no physical change in the good itself.

Certain goods seem to suffer similar losses in desirability through time. Dairy products, for which publicity about their possible relationship to certain circulatory diseases has been increasing through time, might be considered to have suffered a systematic decline in desirability to consumers.

These examples raise the important question of just what we mean by a taste change as opposed to a quality change. To take a slightly different idealized case, suppose that consumers suddenly learn to use a certain fuel more efficiently, getting a certain number of BTUS out of a smaller quantity of fuel. If the relevant axis on the indifference map is the amount of fuel *purchased*, then there has been a taste change; if it is the number of BTUS gained from such fuel, there has not been a taste change but a quality change – a change in the opportunities available to consumers. The change can be consistently treated in either way, but the two treatments will differ. When the phenomenon is treated as a quality change, the true cost-of-living index will decline; when it is analyzed as a taste change, this will not be the case. The decision turns on whether the cost of living should be said to decrease just because consumers are better at consuming. If we are concerned with the delivery to the consumer of certain 'basic satisfactions', a quality change is involved; this is an extension of the position taken in the construction of hedonic price indices. If, on the other hand, we are concerned with the valuation of opportunities *as available in the market*, then treatment of the change as being one of tastes is more appropriate. Both positions are tenable and both can lead to uncomfortable results if pushed to absurdity. (Suppose on the one hand that the new technique is discovered and popularized by fuel sellers. Suppose, on the other, that there is no change in the technology of fuel use but that people decide they now prefer a lower temperature in their houses.) The present section treats taste changes, the quality change case which is similarly parametrizable being treated in section V.

In this section, the case in which taste change may be parametrized as solely good-augmenting is treated in detail. A taste change is said to be good-augmenting if and only if the preference maps can be represented by a utility function whose ith argument is a function of the amount of purchases of the ith good and of the level of some taste change parameter.[9] Following the terminology employed in capital theory, we might call a taste change which is independent of any change in the qualities of the goods a disembodied taste change. In this section, the effect of such taste change upon the value of the true cost-of-living index is studied. We derive results in terms of the parameters of the demand functions which are, in particular, capable of being estimated from market data.

For convenience, assume that only one good, say the first, experiences an own-augmenting disembodied taste change. (Immediate generalization of the results to the case where more than one of the n goods experience own-augmenting disembodied taste changes is discussed at the end of this section.) Let the representative consumer's utility function be given by $u(bx_1,x_2,\ldots,x_n)$, where b is the parameter representing first-good-augmenting taste change and x_i $(i = 1,2,\ldots,n)$ is the amount of the ith good that is purchased.[10] Also assume that $u(\cdot)$ is an increasing, twice differentiable, strictly quasi-concave function which is defined over the nonnegative orthant of an n-dimensional space.[11] For the purposes of this section we also assume that all relevant maxima and minima are given by interior solutions to the first-order conditions. Corner solutions are treated in section IV.

We now turn to the formal analysis of the problem. If with current tastes the representative consumer faces base period income $\hat{y}$ and base period prices $\hat{p}$ where $\hat{p}$ is an n-dimensional column vector defined by $\hat{p}' = (\hat{p}_1,\hat{p}_2,\ldots,\hat{p}_n)$, his purchases would have been given by the column vector $\hat{x}$ which is defined by $\hat{x}' = (\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)$. $\hat{p}_i$ and $\hat{x}_i$ $(i = 1,2,\ldots,n)$ are respectively the base period price of the ith good and the amount of the ith good that *would have been* purchased if he had faced the base period constraints with current tastes. $\hat{x}$ is found by solving the system of first-order conditions:

$$\begin{pmatrix} \hat{p}'\hat{x} \\ b\hat{u}_1 \\ \hat{u}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{u}_n \end{pmatrix} - \begin{pmatrix} \hat{y} \\ \text{---} \\ \hat{\lambda}\hat{p} \end{pmatrix} = 0, \tag{3.1}$$

where $\hat{u}_i(i = 1,2,\ldots,n)$ denotes the derivative of $u(\cdot)$ with respect to its ith argument evaluated at the point $\hat{x}$. $\hat{\lambda}$ is a nonnegative scalar Lagrange multiplier which has the (cardinal) interpretation of the current marginal utility of income when prices are evaluated at $\hat{p}$ and income is $\hat{y}$.

Next we solve for that income y that makes the individual currently indifferent between his current constraints and his base period constraints. y is defined by

$$p'x-y = 0, \tag{3.2}$$

where p is the column vector of current prices, $p' = (p_1,p_2,\ldots,p_n)$, where

p_i $(i = 1,2,\ldots,n)$ is the current price of the ith good. x is the column vector of purchases, $x' = (x_1,x_2,\ldots,x_n)$, that minimizes y subject to $u(bx_1,x_2,\ldots,x_n) = u(b\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)$. Constrained minimization of y implies that

$$\begin{pmatrix} u \\ bu_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} - \begin{pmatrix} \hat{u} \\ \text{---} \\ \lambda p \end{pmatrix} = 0\,, \tag{3.3}$$

where u_i $(i = 1,2,\ldots,n)$ denotes differentiation of $u(\cdot)$ with respect to its ith argument evaluated at x, $\hat{u}$ denotes $u(b\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)$, and λ is a non-negative Lagrange multiplier.

We are interested in how the true cost-of-living index $(y/\hat{y})$ is affected by taste change. Thus, it is necessary to develop the total derivative of y with respect to b. Base-period income $\hat{y}$, base-period prices $\hat{p}$, and current prices p are the given data of the problem. We evaluate $(\partial y/\partial b)$ in steps.

Lemma 3.1. $\left(\dfrac{\partial y}{\partial b}\right)_{u=\hat{u}\text{ const.}} = \dfrac{-p_1x_1}{b}.$

Proof. Total differentiation of (3.3) with respect to b yields:

$$\begin{bmatrix} 0 & bu_1 & u_2 & \ldots & u_n \\ p_1 & b^2u_{11} & bu_{12} & \ldots & bu_{1n} \\ p_2 & bu_{21} & u_{22} & \ldots & u_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ p_n & bu_{n1} & u_{n2} & & u_{nn} \end{bmatrix} \begin{bmatrix} \dfrac{-\partial\lambda}{\partial b} \\ \text{------} \\ \dfrac{\partial x}{\partial b} \\ \\ \\ \end{bmatrix} + \begin{bmatrix} x_1u_1 \\ u_1+bx_1u_{11} \\ x_1u_{12} \\ \cdot \\ \cdot \\ \cdot \\ x_1u_{1n} \end{bmatrix} = 0\,, \tag{3.4}$$

where u_{ij} $(i,j = 1,2,\ldots,n)$ denotes partial differentiation of u_i with respect to its jth argument and $(\partial x/\partial b)$ denotes the column vector $(\partial x_1/\partial b, \partial x_2/\partial b,\ldots,\partial x_n/\partial b)'$. Denote the nonsingular $(n+1)\times(n+1)$ matrix in

(3.4) by H. Then:

$$\begin{bmatrix} \dfrac{-\partial\lambda}{\partial b} \\ \text{------} \\ \dfrac{\partial x}{\partial b} \end{bmatrix} = -H^{-1} \begin{bmatrix} x_1 u_1 \\ u_1 + b x_1 u_{11} \\ x_1 u_{12} \\ \cdot \\ \cdot \\ \cdot \\ x_1 u_{1n} \end{bmatrix}. \tag{3.5}$$

But from (3.2) and (3.3)

$$\left(\frac{\partial y}{\partial b}\right)_{u=\bar{u}\text{ const.}} = p'\left(\frac{\partial x}{\partial b}\right) = (0 \mid p')\begin{pmatrix} \dfrac{-\partial\lambda}{\partial b} \\ \text{------} \\ \dfrac{\partial x}{\partial b} \end{pmatrix}$$

$$= -(0 \mid p')H^{-1}\begin{pmatrix} x_1 u_1 \\ u_1 + b x_1 u_{11} \\ x_1 u_{12} \\ \cdot \\ \cdot \\ \cdot \\ x_1 u_{1n} \end{pmatrix} \tag{3.6}$$

in view of (3.5). By (3.3), the first row in H is equal to λ times $(0 \mid p')$ so by the definition of the matrix inverse we have that

$$\left(\frac{\partial y}{\partial b}\right)_{u=\bar{u}\text{ const.}} = \frac{-x_1 u_1}{\lambda} = \frac{-p_1 x_1}{b} \tag{3.7}$$

by (3.3), which proves the lemma.

Following the practice in capital theory, a fruitful way to understand Lemma 3.1 is to proceed by measuring the purchases of the various goods in (utility) efficiency units. Let x^*, the vector of purchases *measured in efficiency units*, be defined by

$$x^{*\prime} = (x_1^*, x_2^*, \ldots, x_n^*) = (bx_1, x_2, \ldots, x_n). \tag{3.8}$$

Since the corresponding vector of *prices per efficiency unit* is $(p_1/b, p_2, \ldots, p_n)$, income y can be written as

$$y = (p_1/b, p_2, \ldots, p_n)x^* . \tag{3.9}$$

Holding x^* fixed, differentiating (3.9) with respect to b yields

$$\left(\frac{\partial y}{\partial b}\right)_{x^*\ \text{const.}} = \frac{-p_1 x_1}{b} = \left(\frac{\partial y}{\partial b}\right)_{u=\hat{u}\ \text{const.}} \tag{3.10}$$

by Lemma 3.1, if x^* and b are such that the system (3.3) is satisfied. Thus the effect on y along a constant utility surface of a first-order change in the taste parameter b is the same as the effect on y, holding the amount of purchases measured in efficiency units constant, of a first-order change in the taste parameter b.[12]

Now define $(\partial\hat{x}/\partial b)$ to be the column vector with ith entry $(\partial\hat{x}_i/\partial b)$ and let $(\partial\hat{u}/\partial\hat{x})$ be the column vector with ith entry $(\partial u/\partial x_i)$ evaluated at $\hat{x}$.

Lemma 3.2. $\left(\dfrac{\partial\hat{u}}{\partial\hat{x}}\right)'\left(\dfrac{\partial\hat{x}}{\partial b}\right) = 0 .$

Proof. Totally differentiating (3.1) with respect to b yields

$$\begin{bmatrix} 0 & \hat{p}_1 & \hat{p}_2 & \cdots & \hat{p}_n \\ \hat{p}_1 & b^2\hat{u}_{11} & b\hat{u}_{12} & \cdots & b\hat{u}_{1n} \\ \hat{p}_2 & b\hat{u}_{21} & \hat{u}_{22} & \cdots & \hat{u}_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \hat{p}_n & b\hat{u}_{n1} & \hat{u}_{n2} & \cdots & \hat{u}_{nn} \end{bmatrix} \begin{bmatrix} -\dfrac{\partial\hat{\lambda}}{\partial b} \\ \text{------} \\ \dfrac{\partial\hat{x}}{\partial b} \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ \hat{u}_1 + b\hat{x}_1\hat{u}_{11} \\ \hat{x}_1\hat{u}_{12} \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_1\hat{u}_{1n} \end{bmatrix} = 0 . \tag{3.11}$$

The $\hat{u}_{ij}$ $(i,j = 1,2,\ldots,n)$ are the cross partials defined previously but evaluated at $\hat{x}$. Let $\hat{J}$ denote the nonsingular matrix in (3.11). Then from (3.1) and (3.11)

$$\left(\frac{\partial \hat{u}}{\partial \hat{x}}\right)'\left(\frac{\partial \hat{x}}{\partial b}\right) = \hat{\lambda}(0 \mid \hat{p}')\begin{pmatrix} -\dfrac{\partial \lambda}{\partial b} \\ \text{-------} \\ \dfrac{\partial \hat{x}}{\partial b} \end{pmatrix}$$

$$= -\hat{\lambda}(0 \mid \hat{p}')\hat{J}^{-1}\begin{pmatrix} 0 \\ \hat{u}_1 + b\hat{x}_1\hat{u}_{11} \\ \hat{x}_1\hat{u}_{12} \\ \vdots \\ \hat{x}_1\hat{u}_{1n} \end{pmatrix}, \tag{3.12}$$

which equals zero because $(0 \mid \hat{p}')$ is the first row in $\hat{J}$. Lemma 3.2 is an 'envelope theorem' where the change in $\hat{u}$ due to a first order change in b *ceteris paribus* is exactly equal to the change in $\hat{u}$ due to first-order change in b when $\hat{x}$ is allowed to vary optimally (*mutatis mutandis*).

Lemma 3.3. $\left(\dfrac{\partial y}{\partial \hat{u}}\right) = \dfrac{1}{\lambda} > 0$.

Lemma 3.3 taken with Lemma 3.2 has the familiar interpretation that λ is the current marginal utility of income when prices are evaluated at p and income is y.

Proof. Total differentiation of (3.2) with respect to $\hat{u}$ yields

$$H\begin{pmatrix} -\dfrac{\partial \lambda}{\partial \hat{u}} \\ \text{------} \\ \dfrac{\partial x}{\partial \hat{u}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.13}$$

where $(\partial x / \partial \hat{u})$ is an n-dimensional column vector with ith entry $(\partial x_i / \partial \hat{u})$. Differentiating (3.2) with respect to $\hat{u}$ and substituting from (3.13) yields

$$\frac{\partial y}{\partial \hat{u}} = p'\left(\frac{\partial x}{\partial \hat{u}}\right) = (0 \mid p')H^{-1}\begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \frac{1}{\lambda}, \tag{3.14}$$

because the first row in H is equal to $\lambda(0 \mid p')$.

From (3.1)–(3.3), total differentiation of y with respect to the parameter b gives

$$\frac{\partial y}{\partial b} = \left(\frac{\partial y}{\partial b}\right)_{u=\hat{u}\text{ const.}} + \left(\frac{\partial y}{\partial \hat{u}}\right)\left[\hat{x}_1\hat{u}_1 + \left(\frac{\partial \hat{u}}{\partial \hat{x}}\right)'\left(\frac{\partial \hat{x}}{\partial b}\right)\right]. \tag{3.15}$$

Theorem 3.1. $\dfrac{\partial y}{\partial b} = \dfrac{p_1 x_1}{b}\left(\dfrac{\hat{x}_1\hat{u}_1}{x_1 u_1} - 1\right).$

Proof. Substitute the results of Lemmas 3.1–3.3 into equation (3.15) and then simplify by using equations (3.1) and (3.3) to establish the theorem.

Substituting from (3.1) and (3.3), (3.15) can be rewritten as

$$\frac{\partial y}{\partial b} = \frac{\hat{x}_1\hat{u}_1 - x_1 u_1}{\lambda}. \tag{3.16}$$

Notice that the numerator of the RHS of (3.16) is the *ceteris paribus* increase in current utility when facing base-period prices minus the *ceteris paribus* increase in current utility when facing current prices, due to a first order increase in the value of b. By Lemma 3.2, we recognize the numerator of the RHS of (3.16) as the additional compensation in units of utility required to keep the consumer indifferent between base period and current constraints when b changes. Since Lemma 3.3 allows λ the interpretation of the marginal utility of income, the full fraction on the RHS of (3.16) gives the same additional compensation in money units.[13]

Corollary 3.1. If $p = \hat{p}$, then $\left(\dfrac{\partial y}{\partial b}\right) = 0$.

Proof. The corollary is an immediate consequence of Theorem 3.1. The corollary is obvious from consideration of the definition of the true cost-of-living index. After all, if $p = \hat{p}$ then $y = \hat{y}$ for all values of b.

Since we know that $(\partial y/\partial b)$ is zero when current prices equal base-period prices, in order to study the effect of taste change on the true cost-of-living index it is natural to investigate the qualitative behavior of $(\partial y/\partial b)$ when prices are displaced from $\hat{p}$. In particular, we want to derive results concerning the sign of $(\partial y/\partial b)$ for values of p different from $\hat{p}$.

To do this, it is convenient to define $z(p) = x_1 u_1$ and to study the effects of price changes upon $z(p)$.

Lemma 3.4. $\dfrac{\partial u_1}{\partial p_1} = \dfrac{u_1}{\lambda}\dfrac{\partial \lambda}{\partial p_1} + \dfrac{u_1}{p_1}$, and

$$\frac{\partial u_1}{\partial p_i} = \frac{u_1}{\lambda}\frac{\partial \lambda}{\partial p_i} \text{ for } i = 2, \ldots, n\,.$$

Proof. From (3.3) we have that $\dfrac{\partial u_1}{\partial p_i} = \dfrac{1}{b}\dfrac{\partial(\lambda p_1)}{\partial p_i}$ for $i = 1, 2, \ldots, n$. The lemma follows immediately.

Lemma 3.5. $\dfrac{1}{\lambda}\left(\dfrac{\partial \lambda}{\partial p_i}\right) = -\left(\dfrac{\partial x_i}{\partial y}\right)_{p\ \text{const.}}$, for $i = 1, 2, \ldots, n$.

Proof. Total differentiation of (3.3) with respect to p_i yields

$$\begin{pmatrix} \dfrac{-\partial \lambda}{\partial p_i} \\ \text{------} \\ \dfrac{\partial x}{\partial p_i} \end{pmatrix} = H^{-1} \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \lambda \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \tag{3.17}$$

where $\left(\dfrac{\partial x}{\partial p_i}\right)$ is a column vector with ith entry $(\partial x_i / \partial p_i)$. The column vector on the RHS of (3.17) has λ for its $(i+1)$st entry with all other entries zero. Therefore

$$\frac{1}{\lambda}\left(\frac{\partial \lambda}{\partial p_i}\right) = -(1\ 0 \ldots 0)H^{-1} \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \tag{3.18}$$

where the unit in the column vector in (3.18) appears in the $(i+1)$st entry. The LHS of (3.18) is thus shown to be equal to minus the element in the first row and $(i+1)$st column of H^{-1} which in turn is equal to minus the element in the first row and $(i+1)$st column of the matrix J^{-1} where J is defined by

$$J = \begin{bmatrix} 0 & p_1 & p_2 & \dots & p_n \\ p_1 & b^2u_{11} & bu_{12} & \dots & bu_{1n} \\ p_2 & bu_{21} & u_{22} & \dots & u_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ p_n & bu_{n1} & u_{n2} & \dots & u_{nn} \end{bmatrix}.$$

This follows because only the first rows of H and J differ and they only differ by a scalar multiple. Consideration of the evaluation of inverses by the adjoint method shows that except for their first entries the first rows of H^{-1} and J^{-1} must be equal. Substituting J^{-1} for H^{-1} in (3.18) and transposing both sides yields

$$\frac{1}{\lambda}\left(\frac{\partial\lambda}{\partial p_i}\right) = -(0\dots0\,1\,0\dots0)J^{-1}\begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \tag{3.19}$$

because J^{-1} is a symmetric matrix.

If the first equation in the system (3.3) is replaced by equation (3.2) and the resulting system is totally differentiated with respect to y holding prices constant, then we have

$$J\begin{pmatrix} \dfrac{-\partial\lambda}{\partial y} \\ \hline \dfrac{\partial x}{\partial y} \end{pmatrix}_{p\ \text{const.}} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \tag{3.20}$$

where $(\partial x/\partial y)$ is a column vector with ith entry $(\partial x_i/\partial y)$. It follows immediately from (3.20) that

$$\left(\frac{\partial x_i}{\partial y}\right)_{p\ \text{const.}} = (0\ldots0\ 1\ 0\ldots0)J^{-1}\begin{pmatrix}1\\0\\\cdot\\\cdot\\\cdot\\0\end{pmatrix}, \tag{3.21}$$

where the unit in the row vector on the RHS of (3.21) appears in the $(i+1)$st entry. The lemma follows after combining (3.19) and (3.21).

Next define the elasticity of demand for the ith good with respect to the first price by

$$\eta_{i1} = \left(\frac{p_1}{x_i}\right)\left(\frac{\partial x_i}{\partial p_1}\right)_{y\ \text{const.}}$$

for $i = 1,2,\ldots,n$.

Lemma 3.6. If $z(p) = x_1u_1$, then

$$\frac{\partial z}{\partial p_1} = \frac{x_1u_1}{p_1}\{\eta_{11}+1\} \quad \text{and}$$

$$\frac{\partial z}{\partial p_i} = \frac{x_iu_1}{p_1}\eta_{i1}, \quad i = 2,\ldots,n\,.$$

Proof. By Lemmas 3.4 and 3.5

$$\frac{\partial z}{\partial p_1} = u_1\left[\left(\frac{\partial x_1}{\partial p_1}\right)_{u=\bar{u}\ \text{const.}} - x_1\left(\frac{\partial x_1}{\partial y}\right)_{p\ \text{const.}} + \frac{x_1}{p_1}\right] \text{and}$$

$$\frac{\partial z}{\partial p_i} = u_1\left[\left(\frac{\partial x_1}{\partial p_i}\right)_{u=\bar{u}\ \text{const.}} - x_1\left(\frac{\partial x_i}{\partial y}\right)_{p\ \text{const.}}\right], \; i = 2,\ldots,n\,. \tag{3.22}$$

Because substitution effects are symmetric, in (3.22), $(\partial x_1/\partial p_i)_{u=\bar{u}\ \text{const.}}$ can be replaced by $(\partial x_i/\partial p_1)_{u=\bar{u}\ \text{const.}}$. Application of Slutsky's theorem then yields

$$\frac{\partial z}{\partial p_1} = u_1\left[\left(\frac{\partial x_1}{\partial p_1}\right)_{y\ \text{const.}} + \frac{x_1}{p_1}\right] \text{and}$$

$$\frac{\partial z}{\partial p_i} = u_1\left(\frac{\partial x_i}{\partial p_1}\right)_{y\ \text{const.}}, \; i = 2,\ldots,n\,. \tag{3.23}$$

Using the definition of the η_{i1} in (3.23) and rearranging completes the proof of the lemma.

We must now agree on some terminology. We shall call the demand for the first good *price elastic* (*price inelastic*) if $\eta_{11} < (>) -1$. Next, we shall call the ith good a *gross substitute* (*gross complement*) for the first good if $\eta_{i1} > (<) 0$ $(i = 2,\ldots,n)$. Note that this relation is not symmetric; the ith good can be a gross substitute for the first good while the first good is a gross complement for the ith good. This, of course, is due to income effects. The symmetric substitution relationships defined by the substitution terms in the Slutsky equation we shall refer to as those of *net substitutes* or *net complements*.

Theorem 3.2. (A) Suppose $p_i = \hat{p}_i$ for $i = 2,\ldots,n$. If the demand for the first good is price elastic, then $(\partial y/\partial b)$ has the same sign as $(p_1 - \hat{p}_1)$. If that demand is price inelastic, then $(\partial y/\partial b)$ and $(p_1 - \hat{p}_1)$ have opposite signs. If $\eta_{11} = -1$, then $(\partial y/\partial b) = 0$.

(B) Suppose $p_i = \hat{p}_i$ for $i = 1,\ldots,n$ and $i \neq j \neq 1$. If the jth good is a gross complement for the first good, then $(\partial y/\partial b)$ has the same sign as $(p_j - \hat{p}_j)$. If the jth good is a gross substitute for the first good, then $(\partial y/\partial b)$ and $(p_j - \hat{p}_j)$ have opposite signs. If $\eta_{j1} = 0$, then $(\partial y/\partial b) = 0$.

(C) If $p_i = k\hat{p}_i$, $i = 1,2,\ldots,n$, where k is a positive constant, then $(\partial y/\partial b) = 0$.

Proof. (A) and (B) follow directly from Theorem 3.1, Corollary 3.1, and Lemma 3.6.

(C) Totally differentiating z with respect to k yields

$$k\frac{\partial z}{\partial k} = u_1 x_1 + u_1 \sum_1^n p_i \left(\frac{\partial x_i}{\partial p_1}\right)_{y\ \text{const.}} \tag{3.24}$$

by Lemma 3.6 since $k(\partial p_i/\partial k) = p_i$ by hypothesis. But from (3.2), $\sum_1^n p_i \left(\frac{\partial x_i}{\partial p_1}\right)_{y\ \text{const.}} = -x_1$. Theorem 3.2 (C) follows from Theorem 3.1 and Corollary 3.1.

Notice that Theorem 3.2 (A) is a *global* result (i.e. it is a result that holds for all values of p_1) when the sign of $(\eta_{11} + 1)$ is independent of the value of p_1. Likewise, Theorem 3.2 (B) is a global result when the sign of η_{j1} is independent of the value of p_j. Theorem 3.2 (C) is an extension of Corollary 3.1. If current prices are all k times base-period prices then the income that makes the consumer currently indifferent between current constraints and base-period constraints is equal to k times base-period income regardless of the value of b.[14]

Theorem 3.2 has important practical implications and may be interpreted as follows. Suppose first that all prices except the jth are the same in the two periods ($1 \leqslant j \leqslant n$). If tastes did not change ($b = 1$), the only change in the cost-of-living index would be due to the change in the value of the jth price from $\hat{p}_j$ to p_j and would, of course, be in the same direction. Assuming b to be increasing through time, if $(\partial y/\partial b)$ has the same sign as $(p_j - \hat{p}_j)$, the effect of the taste change is to magnify the effect of the change in p_j. One can express this by saying that the jth good ought to receive increased weight in the index because of the taste change. Similarly, if $(\partial y/\partial b)$ and $(p_j - \hat{p}_j)$ have opposite signs, the effect of the taste change reduces the effect of the change in p_j and the jth good ought to receive a decreased weight. Since we can always analyze a change in more than one price (for our purposes) as a series of individual price changes (because of the definition of the true cost-of-living index), these conclusions are not restricted to cases in which only one price changes between the two periods considered. Thus, Theorem 3.2 suggests that in practice, when computing a cost-of-living index, the recently introduced good should receive more weight (less weight) if demand for it is price elastic (price inelastic) than it would in a price index that does not allow for taste change. Similarly the prices of the goods that are gross complements for the recently introduced good should receive more weight and gross substitutes less weight than they would be given in a traditional price index.

Under certain conditions, e.g. homotheticity of the indifference map, we know that the true cost-of-living index $(y/\hat{y})$ is such that

$$\left(\frac{p'x}{\hat{p}'x}\right) \leqslant \left(\frac{y}{\hat{y}}\right) \leqslant \left(\frac{p'\hat{x}}{\hat{p}'\hat{x}}\right) \tag{3.25}$$

because the price indices on the left and the right do not account for substitution effects. The price index on the left of (3.25) is the (current weight) Paasche index. If tastes have not changed, the price index on the right is equal to the (base-period weight) Laspeyres index, since in that case the vector $\hat{x}$ is equal to the vector $\tilde{x}$, an n-dimensional column vector with ith entry $\tilde{x}_i$ denoting the quantity of the ith good actually purchased during the base period. Since the vector $\hat{x}$ is not observed while the vector $\tilde{x}$ is observed, it is of interest to know the relationship of the Laspeyres index $(p'\tilde{x}/\hat{p}'\tilde{x})$ to the unobserved index $(p'\hat{x}/\hat{p}'\hat{x})$. This is the purpose of the next theorem.

Theorem 3.3. (A) $\frac{\partial \hat{x}_1}{\partial b} = \frac{-\hat{x}_1}{b}(1+\eta_{11})$

(B) $\frac{\partial \hat{x}_i}{\partial b} = \frac{-\hat{x}_i}{b}\eta_{i1}, \; i = 2,\ldots,n.$

Proof. Theorem 3.3 can be easily proved by appropriate manipulation of equation (3.11). It is more interesting, however, to analyze the problem when purchases are measured in efficiency units. Let $\hat{x}_1^* = b\hat{x}_1$ be the amount of the first good purchased (measured in efficiency units) when prices are $\hat{p}$. $\hat{p}_1^* = (\hat{p}_1/b)$ is the price per efficiency unit of the first good. The equilibrium amounts of purchases measured in efficiency units depend only upon prices per efficiency unit and income $\hat{y}$. For $\hat{p}_1^*$ fixed, the amounts of equilibrium purchases are independent of the values of b and $\hat{p}_1$. Therefore we conclude that

$$\left(\frac{\partial \hat{x}_1^*}{\partial b}\right)\left(\frac{\partial b}{\partial \hat{p}_1^*}\right)_{\hat{p}_1 \text{ const.}} = \left(\frac{\partial \hat{x}_1^*}{\partial \hat{p}_1}\right)\left(\frac{\partial \hat{p}_1}{\partial \hat{p}_1^*}\right)_{b \text{ const.}} \tag{3.26}$$

and

$$\left(\frac{\partial \hat{x}_i}{\partial b}\right)\left(\frac{\partial b}{\partial \hat{p}_1^*}\right)_{\hat{p}_1 \text{ const.}} = \left(\frac{\partial \hat{x}_i}{\partial \hat{p}_1}\right)\left(\frac{\partial \hat{p}_1}{\partial \hat{p}_1^*}\right)_{b \text{ const.}}, \; i = 2,\ldots,n. \tag{3.27}$$

Using the definitions of $\hat{x}_1^*$, $\hat{p}_1^*$, and η_{11} in (3.26) yields (A). Using the definitions of p_1^* and η_{i1} in (3.27) yields (B). The price elasticities of demand η_{i1}, $i = 1,2,\ldots,n$, in (A) and (B) are evaluated at $\hat{p}$, $\hat{x}$, and $\hat{y}$.

Again consider the case in which the first good has been recently introduced and thus the value of b has been increasing through time. Theorem 3.3 tells us, e.g. that if the price of the recently introduced good has fallen ($\hat{p}_1 > p_1$) and the demand for the first good is price elastic while the prices of all goods that are gross complements (gross substitutes) for the first good are falling (rising), then $(p'\tilde{x}/\hat{p}'\tilde{x}) > (p'\hat{x}/\hat{p}'\hat{x})$. In this special case, therefore, the value of the true cost-of-living index lies between the values of the Paasche and Laspeyres price indices (subject, of course, to the qualifications discussed in footnote 7). This result can also be deduced from Theorem 3.2 because in this special case $(\partial y/\partial b) < 0$.

Theorem 3.3 reinforces Theorem 3.2. It tells us that had current tastes been in force during the base period, purchases of gross complements for the recently introduced good would have been greater and purchases of gross substitutes less than was actually the case. Similarly, the demand for the good itself would have been greater (less) if its demand is price elastic.

It follows that in constructing a Laspeyres price index, the price of the recently introduced good should receive more weight (less weight) if demand for it is price elastic (price inelastic). Similarly the prices of goods that are gross complements (gross substitutes) for the recently introduced good should receive more weight (less weight). Theorem 3.2 assures us that similar weight changes should be made in a true cost-of-living index (a Paasche index, of course, needs no such corrections).[15]

We have stated Theorem 3.2 (and interpreted Theorem 3.3) in qualitative terms to give them some practical usefulness. In practice, one might very well be willing to say that a taste change of the sort described (a change in b) has occurred, but it is unlikely that one would be willing to say by how much b has changed. Obviously, if such information were somehow available, our lemmas would yield precise quantitative results.

Theorems 3.1–3.3 can be extended to include cases where more than one good has experienced an own-augmenting taste change. For example, consider the case in which the first two goods have been recently introduced so that the preference maps can be represented by the utility function $u(b_1x_1, b_2x_2, x_3, \ldots, x_n)$ where b_1 and b_2 have been increasing through time. In constructing a cost-of-living index, prices of those goods that are gross complements (gross substitutes) for *both* of the recently introduced goods should receive more weight (less weight). If demand for the first good is elastic and demand for the second is inelastic, if the first and second goods are gross substitutes for each other, and if $(b_1/b_2) > 1$ in the current period while $(b_1/b_2) = 1$ during the base period, then the first good should receive more weight and the second good less weight.

Before closing this section, we may briefly ask a second-order question. Do the effects described in Theorem 3.2 get larger with larger price changes or do they decrease as price changes increase? This question is of some interest if attention is to be paid to such effects in practice. Since, as in Theorem 3.2, it suffices to look at one price change at a time, we may answer it by examining $(\partial^2 y/\partial b\, \partial p_j)$ $(j = 1, \ldots, n)$.

Define the *net* price elasticity of demand η_{j1}^n by

$$\eta_{j1}^n = \left(\frac{p_j}{x_1}\right)\left(\frac{\partial x_1}{\partial p_j}\right)_{u=\hat{u}\,\text{const}}, \quad j = 1, \ldots, n. \tag{3.28}$$

Lemma 3.7. $$\frac{\partial^2 y}{\partial b \partial p_1} = \left(\frac{1}{p_1}\right)\left(\frac{\partial y}{\partial b}\right)\{\eta_{11}^n + 1\} - \left(\frac{\hat{x}_1 \hat{u}_1}{b u_1}\right)(\eta_{11} + 1)$$

and
$$\frac{\partial^2 y}{\partial b \partial p_j} = \left(\frac{1}{p_j}\right)\left(\frac{\partial y}{\partial b}\right)\eta^n_{j1} - \left(\frac{\hat{x}_1 \hat{u}_1 x_j}{b x_1 u_1}\right)\eta_{j1} (j = 2, \ldots, n).$$

Proof. This follows immediately from Theorem 3.1 and Lemma 3.6.

We may now state:

Theorem 3.4. (A) Suppose that $p_i = \hat{p}_i$, $i = 2, \ldots, n$. For p_1 sufficiently close to $\hat{p}_1$, $(\partial^2 y / \partial b \, \partial p_1)$ is positive if the demand for the first good is elastic and negative if it is inelastic. Further, if $\eta^n_{11} \geqslant -1$, the same statement holds for all $p_1 > \hat{p}_1$; if $\eta^n_{11} \leqslant -1$, it holds for all $p_1 < \hat{p}_1$.[16]

(B) Suppose that $p_i = \hat{p}_i$ for $i = 1, \ldots, n$ and $i \neq j \neq 1$. For p_j sufficiently close to $\hat{p}_j$, $(\partial^2 y / \partial b \, \partial p_j)$ is positive if the jth good is a gross complement for the first good and negative if the jth good is a gross substitute for the first good. Further, if the two goods are *net* substitutes (or if $\eta^n_{j1} = 0$), the same statement holds for all $p_j > \hat{p}_j$; if they are *net* complements (or if $\eta^n_{j1} = 0$), it holds for all $p_j < \hat{p}_j$.

Proof. The statements about sufficiently small price changes follow from Lemma 3.7 and Corollary 3.1. The remaining statements follow from Lemma 3.7 and Theorem 3.2.

Thus, for all cases which can be definitely determined, the second-order effects being examined reinforce the first-order ones already treated. The effects of taste change on proper weights in the cost-of-living index are bigger for bigger price changes. For example, we have already seen in Theorem 3.2 that the weight given a gross complement for the first good should be increased on account of the taste change. We now see that for small changes in the price of that complement this effect gets bigger the bigger the price change, and that this remains true globally if the goods are also net complements and the price of the good in question has fallen. Similarly, if the jth good is a gross substitute for the first good, the weight given the jth good should be decreased as a result of the taste change. The amount of decrease should be greater, the higher is p_j above $\hat{p}_j$, provided that the two goods are net substitutes as well.[17]

IV. *New Goods and Other Corner Solutions*

In the previous section, we restricted our analysis of taste change to cases where the relevant maxima and minima are given by interior solutions to the first-order conditions. This section is devoted to a general analysis of the treatment of corner solutions in the cost-of-living index. The problem of this type that is most frequently encountered in practice is the problem of 'new goods'. For our purposes, a new good is one that is purchased in positive amount during the current period but for which base-period purchases were zero. The opposite case of 'disappearing goods', where purchases of the disappearing goods were positive in the base period but are zero in the current period, is also of practical interest.

Using the vector form of the notation developed in (2.1)–(2.4), the problem is to find that income y that makes the representative consumer currently indifferent between facing current prices p with income y and facing base-period prices $\hat{p}$ and base-period income $\hat{y}$. Formally the problem is to solve for a nonnegative vector of purchases x such that

$$\left(\frac{\partial u}{\partial x}\right)-\lambda p\leqslant 0\,, \tag{4.1}$$

where $(\partial u/\partial x)$ is a column vector with ith entry $(\partial u/\partial x_i)$, $i = 1,2,\ldots,n$,

$$x'\left[\left(\frac{\partial u}{\partial x}\right)-\lambda p\right]=0\,, \tag{4.2}$$

$$x\geqslant 0 \quad \text{and} \quad \lambda\geqslant 0\,. \tag{4.3}$$

x is constrained by $u(x) = u(\hat{x})$ or simply

$$u-\hat{u} = 0\,, \tag{4.4}$$

where $\hat{x}$ solves the system

$$\left(\frac{\partial \hat{u}}{\partial \hat{x}}\right)-\hat{\lambda}\hat{p}\leqslant 0\,, \tag{4.5}$$

where $(\partial\hat{u}/\partial\hat{x})$ denotes the vector $(\partial u/\partial x)$ evaluated at $\hat{x}$,

$$\hat{x}'\left[\left(\frac{\partial \hat{u}}{\partial \hat{x}}\right)-\hat{\lambda}\hat{p}\right]=0\,, \tag{4.6}$$

$$\hat{\lambda}(\hat{y}-\hat{p}'\hat{x}) = 0, \tag{4.7}$$

$$\hat{x}\geqslant 0 \quad \text{and} \quad \hat{\lambda}\geqslant 0\,. \tag{4.8}$$

Income y is defined by

$$y-p'x = 0\,, \tag{4.9}$$

and $(y/\hat{y})$ is the true cost-of-living index.[18]

Inequation (4.1) and equation (4.2) imply that if for any $k = 1, 2, \ldots, n$, $(\partial u/\partial x_k) < \lambda p_k$, then $x_k = 0$. A similar implication is drawn from (4.5) and (4.6). λ and $\hat{\lambda}$ are scalar Lagrange multipliers. In (4.7), if we assume nonsatiation in consumption then the budget constraint holds with equality.

Now assume that the kth good is a new good; that is, $x_k > 0$ with $(\partial u / \partial x_k) = \lambda p_k$ and $\tilde{x}_k = 0$, where $\tilde{x}_k$ is the *actual* amount of the kth good that was purchased during the base period. If tastes have not changed, then $\hat{x}_k = \tilde{x}_k = 0$. The difficulty in this case is that there is no recorded base-period market price for the kth good. In the case of no taste change, the computation of the true cost-of-living index which allows for corner solutions is straightforward. If, for example, the kth good is a new good, the restriction $\hat{x}_k = 0$ is added to the system (4.1)–(4.9) leaving the value of $\hat{p}_k$ as an unknown to be determined in solving the new system. Or equivalently, the system (4.1)–(4.9) is solved for y after assigning to $\hat{p}_k$ any value greater than or equal to the demand reservation price (the lowest price at which demand for the kth good is zero) including the supply reservation price (the highest price at which supply of the kth good is zero) which in some sense is the price that consumers actually faced during the base period.

Note, however, that in the base-period constrained utility maximization problem, the demand reservation price itself is the maximizing value of the shadow multiplier associated with the constraint $\hat{x}_k = 0$, since by definition the demand reservation price is what the representative consumer is willing to pay per unit (locally) for a relaxation of the constraint $\hat{x}_k = 0$.[19]

As stated in section II, it is a well-known proposition in the traditional theory of index numbers (v. Hofsten, 1952, pp. 28–9) (where it is assumed that tastes and qualities are unchanging and that all goods are purchased in positive amounts) that under certain conditions the Laspeyres (base-period weighted) price index $(p'\hat{x}/\hat{p}'\hat{x})$ bounds the true cost-of-living index from above, while the Paasche (current-period weighted) price index $(p'x/\hat{p}'x)$ bounds the true cost-of-living index from below. In the case with new goods, it is obvious that the Laspeyres index bounds the true index from above and is independent of the assignment of base-period price weights to the new goods. If we allow for the complication of new goods, however, the Paasche price index is a lower bound upon the true cost-of-living index only if we assign to the new goods, base-period prices greater than or equal to the demand reservation prices. Note, however, that of all such Paasche indices the largest (and therefore in a sense the greatest lower bound on the true cost-of-living index) is the index in which new goods purchases are weighted by their demand reservation prices. (The analysis for disappearing goods is similar and is left to the reader.)

Thus, if they are known, it is the demand reservation prices themselves which should be used to weight new-goods purchases in the construction of a Paasche index and not simply some arbitrary prices equal to or greater than the demand reservation prices. In particular, *supply* reservation prices are not relevant if the demand reservation prices are known.

This is a natural result if we recall that the demand reservation price measures (locally) the value to the base-period consumer of the relaxation of the constraint stating that the good in question is unavailable. It is the shadow price of that constraint. It is thus the demand reservation price which affects how much income the consumer would be willing to give up to relax that constraint. How much income he would in fact be technologically required to give up to accomplish such relaxation (the supply reservation price) is not directly germane to a theory which runs in terms of indifferent positions. If the demand reservation price is known, the supply reservation price is not relevant.

There remains the difficult practical question as to how one knows the values of demand reservation prices. To ascertain them in general might require a rather detailed demand analysis which might not be available. There are some special circumstances, however, in which demand reservation prices may be less difficult to determine. Suppose that it was known that during a period for which closely spaced, time-series data are available the supply reservation price of a certain good is falling. With constant tastes and qualities and all other prices constant, the price at which the good was first marketed would then be the demand reservation price. Also, since the supply reservation price is never less than the demand reservation price, supply reservation prices can be used for new goods in the Paasche index and the latter will retain its property as lower bound on the true index (but see footnote 7).

In order to study the effects of new goods on the true cost-of-living index when tastes are changing, the previous analysis can be combined with the analysis of section III. If, for example, the first good is a new good that has experienced a positive own-augmenting taste change, if the price of the first good has fallen while all other prices have remained constant, and if demand for the first good is elastic, then by Theorems 3.2 and 3.3 the value of the true cost-of-living index is below the value of the Laspeyres index for whatever base-period prices are assigned to the new good. The Paasche index is known to be a lower bound for the true cost-of-living index if and only if the new good is assigned a base-period price greater than or equal to its demand reservation price (subject to the qualification discussed in footnote 8).

v. *Quality Change*

In this section, we take up the problem of quality change.[20] In practice, quality change is handled in the consumer price index (when it is handled at all) by assuming that an improvement in quality in a given good is equivalent to a price reduction in that good. For some cases of quality change, this is obviously the appropriate general treatment. If widgets are sold by the box and twenty widgets now are packed into the same size box as previously held ten, it is clear that this is equivalent to a halving of the price of widgets. Somewhat more generally, if one new widget delivers the same services as two old ones, this may also be considered to be simply a repackaging of widgets and thus equivalent to a price reduction.

Quality change may take other forms than that of simply augmenting the services of just that good whose quality has changed, however, and a simple adjustment of the price of that good may not suffice to account for that quality change in a cost-of-living index. Indeed, we show that such a price adjustment made independently of the amount of all goods purchased is an appropriate one if *and only if* the only effect of quality change is of the good-augmenting type just considered. Then and only then can quality change be considered a simple repackaging of the good in question.

Furthermore, while an adjustment in the price of the quality-changing good can always be made to suffice *locally* (that is, for given purchases of all goods), in general, the price adjustment which must be made will depend on all prices and purchases of all commodities and not simply on the physical characteristics of the quality change. If the new and the old qualities of the good sell in positive amount on the same (perfect) market, then all the information needed to make the appropriate *local* price adjustment for the quality change is of course coded in the difference in the prices of the two varieties. The extension of the same price adjustment to other (perhaps later) situations, however, when other prices change or other related qualities are introduced is appropriate, as stated, only in the pure repackaging case. If the two varieties do not coexist in the same (perfect) market, then even such a local price adjustment must be made to depend explicitly on the quantities of all goods purchased and not simply on physical characteristics, save in the pure repackaging case.[21]

In circumstances other than the simple repackaging case, then, we show that the simplest adjustment of the cost-of-living index may be an adjustment in the price of one or more goods *other than the one whose quality has changed.* While part of the effects of any quality change may well be to augment the services of the quality-changing good, there are likely to be other effects as well and here more than one price change is required.

Thus, for example, suppose that there is a quality change in refrigerators. If this change simply makes one new refrigerator deliver the services of some larger number of old ones, then the simplest price adjustment in the cost-of-living index is indeed an adjustment in the price of refrigerators. On the other hand, if that quality change also increases the enjoyment obtained from a quart of ice cream, then an adjustment in refrigerator price will not suffice; an adjustment in the price of ice cream is also called for. Indeed, if the *only* effect of a refrigerator quality change is to augment the enjoyment obtained from ice cream, then the simplest adjustment is one made *only* in the price of ice cream, even though the quality change takes place in refrigerators. In this case, an adjustment in the price of refrigerators can be made to suffice; the magnitude of that adjustment, however, will depend on the quantities demanded of all goods. An adjustment in the price of ice cream will also suffice; the magnitude of that adjustment, however, will only depend on the quantity of ice cream and the quantity of refrigerators.

Now, of course, this is fairly easy to see in the case of this example. Refrigerators are not directly consumed, rather, they are used as an intermediate good in the production of certain consumption goods, including cold ice cream. Thus, one can argue, since refrigerator services do not enter the utility function directly, the cost of using refrigerator services is but part of the price of the foodstuffs concerned and an improvement in refrigerator quality ought clearly to be accounted for in the prices of just those particular foodstuffs affected. If that quality improvement only changes ice cream enjoyment, then the true quality improvement is in refrigerated ice cream. An adjustment in the price of refrigerated ice cream, however, is most easily done by adjusting the price of ice cream (assuming all ice cream to be refrigerated); an adjustment in the price of refrigerators, on the other hand, affects the cost of consuming other refrigerated foodstuffs as well. Thus, in this simple example, adjustment of the price of ice cream can be made much more simply than adjustment of the price of refrigerators to achieve the same result in the cost-of-living index.

In fact, this is quite a good way to look at the matter and at our results even if refrigerator services do appear in the utility function directly, as is the case in some treatments [22] and as would certainly be the analogous case in treatments of other examples. In this case, refrigerators should *still* be looked on as an intermediate good, affecting the enjoyment of foodstuffs and also the enjoyment of its own services. As before, it is those 'final' goods whose enjoyment is affected by the quality change whose prices should be adjusted to obtain the simplest equivalent change in the cost-of-living index. The fact that one of those 'final' goods happens to have the same name and to be consumed in fixed proportions with the intermediate good does not change this statement. If this is borne in mind throughout, the interpretation of our results will be relatively straightforward.

We now turn to the formal analysis of the problem. The current (twice differentiable) utility function is given by

$$u = u(x_1, \ldots, x_n, b) \equiv u(x, b), \tag{5.1}$$

where b is a parameter measuring quality change in the first good, with $b = 1$ being the case of no quality change.[23] As quality change is to take place in the first good, it is natural to assume

$$u_b(0, x_2, \ldots, x_n, b) \equiv 0, \tag{5.2}$$

where the subscript denotes differentiation with respect to b. However, we shall not make direct use of this property.[24]

As before, in the base period, the consumer has income $\hat{y}$ and faces prices $\hat{p}$. He is also constrained in that period by only being able to purchase a quality of the first commodity for which $b = 1$. The purchases which are made under these conditions are $\hat{x}$, and the corresponding utility level is

$$\hat{u} = u(\hat{x}, 1). \tag{5.3}$$

The constraints of the present period are defined by some $b \neq 1$ and prices p. The income at which the consumer would be just indifferent between the two sets of constraints is y, and the true cost-of-living index is $y/\hat{y}$. y is thus defined as

$$y = p'x\,; \tag{5.4}$$

where x is given as the solution to the problem

$$\text{Minimize } y \text{ subject to } u(x,b) = \hat{u}\,. \tag{5.5}$$

x thus satisfies:

$$u(x,b) - \hat{u} = 0 \tag{5.6}$$

$$u_i - \lambda p_i = 0 \qquad (i = 1,\ldots,n)\,,$$

where λ is a Lagrange multiplier and is the marginal utility of income.[26]

Given $\hat{p}$ and $\hat{y}$, therefore, y is a function of p and b, and we may write

$$y = y(p,b)\,. \tag{5.7}$$

Suppose now that we wish to take account of the quality change by a suitable change in the price of the first good. We thus seek a p_1^*, such that

$$y(p_1^*, p_2, \ldots, p_n, 1) = y(p_1, \ldots, p_n, b)\,. \tag{5.8}$$

For $b = 1$, $p_1^* = p_1$. As b changes from unity, p_1^* will change. Differentiating (5.8) totally with respect to b and rearranging, we have:

$$\frac{\partial p_1^*}{\partial b} = \frac{\partial y/\partial b}{\partial y/\partial p_1^*}\,. \tag{5.9}$$

We must therefore investigate $\partial y/\partial b$ and $\partial y/\partial p_1^*$.

Lemma 5.1. $\partial y/\partial b = -u_b/\lambda$.

Proof. Differentiate (5.6) totally with respect to b, obtaining

$$\begin{bmatrix} 0 & u_1 & \ldots & u_n \\ p_1 & u_{11} & \ldots & u_{1n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ p_n & u_{n1} & \ldots & u_{nn} \end{bmatrix} \begin{bmatrix} -\partial\lambda/\partial b \\ \hline \partial x/\partial b \\ \\ \\ \\ \end{bmatrix} = -\begin{bmatrix} u_b \\ u_{1b} \\ \cdot \\ \cdot \\ \cdot \\ u_{nb} \end{bmatrix}, \tag{5.10}$$

where $\partial x/\partial b$ is an n-component vector whose ith element is $\partial x_i/\partial b$.

Denote the first matrix on the left by D. Then

$$\begin{bmatrix} -\partial\lambda/\partial b \\ \hdashline \partial x/\partial b \end{bmatrix} = -D^{-1}\begin{bmatrix} u_b \\ u_{1b} \\ \cdot \\ \cdot \\ \cdot \\ u_{nb} \end{bmatrix}. \tag{5.11}$$

Now,

$$\begin{aligned} \partial y/\partial b = p'(\partial x/\partial b) &= (0 \mid p')\begin{pmatrix} -\partial\lambda/\partial b \\ \hdashline \partial x/\partial b \end{pmatrix} \\ &= \frac{1}{\lambda}(0,u_1,\ldots,u_n)\begin{pmatrix} -\partial\lambda/\partial b \\ \hdashline \partial x/\partial b \end{pmatrix} \end{aligned} \tag{5.12}$$

in view of (5.6).

However, $(0,u_1,\ldots,u_n)$ is the first row of D and the lemma now follows immediately from (5.11) and (5.12).[26]

Lemma 5.2. $\partial y/\partial p_1 = x_1$.

Proof. Differentiate (5.6) totally with respect to p_1, obtaining:

$$\begin{pmatrix} -\partial\lambda/\partial p_1 \\ \hdashline \partial x/\partial p_1 \end{pmatrix} = D^{-1}\begin{bmatrix} 0 \\ \lambda \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \tag{5.13}$$

where $\partial x/\partial p_1$ is the n-component vector whose ith element is $\partial x_i/\partial p_1$.

$$\partial y/\partial p_1 = x_1 + p'(\partial x/\partial p_1) = x_1 + \frac{1}{\lambda}(0,u_1,\ldots,u_n)\begin{pmatrix} -\partial\lambda/\partial b \\ \hdashline \partial x/\partial b \end{pmatrix}. \tag{5.14}$$

The lemma now follows as before, since $(0,u_1,\ldots,u_n)$ is the first row of D.

Thus, $\partial y/\partial p_1 = x_1$. Similarly, if we substitute p_1^* for p_1 and write x_1^* for the corresponding amount of the first commodity purchased, $\partial y/\partial p_1^* = x_1^*$. It is thus clear that as long as $x_1^* \neq 0$, p_1^* is a uniquely defined function of

b (given the other elements of p). Since at $b = 1, p_1^* = p_1$ no matter what the values of the other elements of p and the elements of x are, p_1^* will be independent of any subset of those elements if and only if $\partial p_1^*/\partial b$ is so independent. We therefore concentrate on the latter quantity. To avoid a burdensome notation, we always take that derivative at $b = 1$; only notational changes would be required to perform the analysis at an arbitrary b.

Combining Lemmas 5.1 and 5.2 with (5.9) and evaluating at $b = 1$, we have

Lemma 5.3: $\dfrac{\partial p_1^*}{\partial b} = \dfrac{-p_1 u_b}{x_1 u_1}$.

Proof. This follows immediately from the two preceding lemmas and (5.6).

Thus we have evaluated the adjustment which must be made in p_1 to give a result equivalent to the quality change involved in a change in b. Clearly, such an adjustment can be made (as long as $x_1 \neq 0$). That adjustment depends in general, however, on all the elements of x. Thus, in the general case, the adjustment cannot be made independent of knowledge of all purchases and the way they affect (u_b/u_1).[27] It is natural to ask under what circumstances the adjustment can be made without such knowledge or, equivalently, under what circumstances an adjustment made from market data in a given situation will retain validity when that situation changes.

Theorem 5.1. (A) A necessary and sufficient condition for $\partial p_1^*/\partial b$ to be independent of $x_2, \ldots, x_n$ is that it be possible to write the utility function in the form

$$u(x,b) = F(g(x_1,b),x_2,\ldots,x_n) \equiv F(g^*(x_1,b)x_1,x_2,\ldots,x_n) \tag{5.15}$$

for some choice of continuously differentiable functions F and g.[28] We write $g(x_1,b) \equiv g^*(x_1,b)x_1$ for ease of interpretation.

(B) A necessary and sufficient condition for $\partial p_1^*/\partial b$ to be independent of *all* the elements of x (including x_1) is that (5.15) hold with g in the form

$$g(x_1,b) = x_1 h(b) \quad \text{or} \quad g^*(x_1,b) = h(b) \tag{5.16}$$

for some choice of the function h. (This is the pure repackaging case.)

Proof. (A) By Lemma 5.3, a necessary and sufficient condition for $\partial p_1^*/\partial b$

to be independent of $x_2, \ldots, x_n$, is that u_b/u_1 be so independent. This is equivalent to (5.15) by a well-known theorem of Leontief (1947a, p. 364; 1947b).

(B) In view of Lemma 5.3 and (A), a necessary and sufficient condition for $\partial p_1^*/\partial b$ to be independent of all the elements of x is that (5.15) hold and that, in addition, $\dfrac{u_b}{x_1 u_1}$ be independent of x_1. This means that it is necessary and sufficient that there exist a function $\phi(b)$ such that

$$\frac{g_b}{g_1} = \frac{u_b}{u_1} = -x_1\phi(b)\,. \tag{5.17}$$

Now consider a curve in the $x_1 - b$ plane along which g is constant–an indifference curve of g. This is defined by

$$g(x_1, b) = \bar{g}\,. \tag{5.18}$$

Differentiating (5.18) totally with respect to b and rearranging:

$$dx_1/db = -\frac{g_b}{g_1} = x_1\phi(b) \tag{5.19}$$

along that curve. Thus:

$$d\log x_1 = \phi(b)\,db\,. \tag{5.20}$$

Integrating:

$$\log x_1 = \log \mu(b) + \log c\,, \tag{5.21}$$

where $\mu(b)$ is an integral of $\phi(b)$, and c is an arbitrary constant. In other words:

$$\frac{x_1}{\mu(b)} = c \tag{5.22}$$

is the equation of the indifference curve defined in (5.18).

Now, we can clearly replace g in (5.15) by any monotonic transformation of $\bar{g}$, adjusting the result by redefining F. Thus we can choose the scale on which g is measured and can do so in such a way as to make $\bar{g} = c$ without changing anything else. If we do this, however, the theorem follows immediately from (5.18) and (5.22), with $h(b) = 1/\mu(b)$.

Some remarks on the theorem are now in order.

First, as observed, part (B) of the theorem is the repackaging case. In this case, it might appear more natural to have b appearing in place of $h(b)$. $h(b)$ appears because the scaling of b is arbitrary. There is no reason not to measure quality change in this case in units of h rather than in units of b, in which case the more natural-appearing result is obtained.

Second, part (B) shows that the repackaging case is the *only* case in which the quality change is equivalent to a *simple* adjustment in the price of the first commodity. Any other case requires knowledge of the elements of x. Another way of putting this is to say that in any other case the adjustment in p_1 will be different at different points in the commodity space.

Third, part (A) shows that, even if we are willing to let the adjustment in p_1 depend on the quantity of the first good purchased, the class of quality changes in the first good which can be so handled is not really much widened. The only generalization is, in effect, to move to a sort of variable repackaging in which the amount of repackaging is allowed to depend on x_1. As soon as a quality change in the first commodity enters in a more general way–for example, by affecting other commodities–an equivalent adjustment in p_1 depends on other elements of x.[29]

Finally, if the conditions of part (A) hold, the dependence of the adjustment on the level of p_1 is of a very simple kind, given x_1. The *percentage* adjustment in p_1 which must be made is dependent only on x_1 in this case, since, given x_1, p_1 enters only multiplicatively in $\partial p_1^* / \partial b$. A similar remark applies to all later results in this section.

Theorem 5.1 can be generalized to give the conditions under which quality change is equivalent to an adjustment in p_1 which depends only on selected elements of x. Thus

Theorem 5.2. (A) For any $m = 1, \ldots, n-1$, a necessary and sufficient condition for $\partial p_1^* / \partial b$ to be independent of $x_{m+1}, \ldots, x_n$, is that it be possible to write the utility function in the form:

$$\begin{aligned} u(x,b) &= F(g(x_1, \ldots, x_m, b), x_2, \ldots, x_n) \\ &\equiv F(g^*(x_1, \ldots, x_m, b) x_1, x_2, \ldots, x_n) \end{aligned} \tag{5.23}$$

for some choice of continuously differentiable functions F and g.[30]

(B) For any $m = 1, \ldots, n-1$, a necessary and sufficient condition for $\partial p_1^* / \partial b$ to be independent of x_1 and $x_{m+1}, \ldots, x_n$ is that (5.23) hold with g in the form

$$g(x_1, \ldots, x_m, b) = x_1 h(x_2, \ldots, x_m, b)$$

or $$g^*(x_1, \ldots, x_m, b) = h(x_2, \ldots, x_m, b), \tag{5.24}$$

for some choice of a function h.

Proof. The proof of part (A) follows again from Leontief's theorem. That of part (B) is the same as that given for part (B) of Theorem 5.1, save that the indifference variety of g is taken at fixed values of $x_2, \ldots, x_m$. The values of $x_2, \ldots, x_m$ then become parameters of $\mu(b)$.

Unfortunately, while this generalization allows us to handle a wider variety of quality change than that covered in Theorem 5.1, it still leaves us in the case of repackaging of the first commodity (although the extent of repackaging is now allowed to depend on the quantities of other commodities). It does not touch the case in which a quality change in the first commodity affects other commodities by augmenting their services, for example, the case of refrigerators and ice cream mentioned above being a case in point. This leads us to abandon the notion that simple adjustments in the price of the good whose quality has changed are likely to be generally effective and to ask whether for some quality changes adjustments in *other* prices might not be more appropriate.

Accordingly, we next examine an extreme case in which only an adjustment in the price of the second commodity is called for. There is an asymmetry in the problem. It was reasonable to ask under what conditions an adjustment in p_1 can be made independent of x_2; it is not reasonable to ask under what conditions an adjustment in p_2 can be made independent of x_1. The quality change is embodied in the first commodity and the consumer cannot take advantage of it without purchasing that commodity (see (5.2), for example). It is reasonable to ask under what circumstances an adjustment in p_2 can be made independent of the other elements of x, however, and this we shall do.

We thus replace (5.8) by

$$y(p_1,p_2^*,p_3,\ldots,p_n,1) = y(p_1,\ldots,p_n,b)\,. \tag{5.25}$$

It is clear that the argument leading to Lemma 5.3 shows

Lemma 5.4. $\dfrac{\partial p_2^*}{\partial b} = \dfrac{-p_2 u_b}{x_2 u_2}$.

We have immediately:

Theorem 5.3. (A) A necessary and sufficient condition for $\partial p_2^* / \partial b$ to be independent of $x_3, \ldots, x_n$ is that it be possible to write the utility function in the form

$$\begin{aligned} u(x,b) &= F(x_1,g(x_1,x_2,b),x_3,\ldots,x_n) \\ &\equiv F(x_1,g^*(x_1,x_2,b)x_2,x_3,\ldots,x_n) \end{aligned} \tag{5.26}$$

for some choice of continuously differentiable functions F and g.[31]

(B) A necessary and sufficient condition for $\partial p_2^* / \partial b$ to be independent of $x_3, \ldots, x_n$ *and* x_2 is that (5.26) hold, with g in the form

$$g(x_1,x_2,b) = x_2 h(x_1,b) \quad \text{or} \quad g^*(x_1,x_2,b) = h(x_1,b) \tag{5.27}$$

for some choice of a function h.

Proof. (A) follows from Lemma 5.4 and Leontief's theorem. (B) is proved as before, noting that x_1 is a parameter of the appropriate indifference curve of g in the $x_2 - b$ plane.

This is an interesting case. Whereas what was interesting about Theorem 5.1 was the necessity of the conditions, what is interesting here is sufficiency. Looked at in this way, the theorem tells us that if quality change in good one augments the services of good *two*, then a simple adjustment in the price of the latter good is called for. Once again, an adjustment can be made in this case in the price of good one, but Theorem 5.1 assures us that the adjustment will not be a simple one; it will depend on all commodity purchases. The simple adjustment is one in the price of the second good which is not the good whose quality has changed. If the only effect of a quality change in refrigerators is to make ice cream taste better, the simple adjustment which should be made is in the price of ice cream, not the price of refrigerators. The magnitude of that adjustment will depend on the quantity of refrigerators, and it may also depend on the quantity of ice cream (which is reasonable when one supposes that the effect depends on the ice cream–refrigerator ratio), but, unlike an adjustment in the price of refrigerators, it does not depend on the quantities of other goods.

Such polar cases, however, are too simple. In practice, quality change, even if it takes the relatively simple form of augmenting the services of certain goods, is unlikely merely to augment the services of only one good. A better refrigerator affects goods other than ice cream. Clearly, from Theorem 5.1 and 5.3, a simple adjustment in a single price will not suffice in such circumstances.

Fortunately, however, simple adjustments in more than one price will suffice, and this can be done by using our results simultaneously for more than one good. Thus, suppose that the utility function can be written in the form

$$\begin{aligned} u(x,b) &= F(g^1(x_1,b),g^2(x_1,x_2,b),\ldots,g^n(x_1,x_n,b)) \qquad (5.28) \\ &\equiv F(g^{*1}(x_1,b)x_1,g^{*2}(x_1,x_2,b)x_2,\ldots,g^{*n}(x_1,x_n,b)x_n) \end{aligned}$$

for some choice of continuously differentiable functions, F and $g^1, \ldots, g^n$. This is the case in which every good is augmented, but, if $g^i(x_1, x_i, b) = x_i$ $(g^{*i}(x_1, x_i, b) = 1)$ for all b, then the augmentation of the ith good is zero (and similarly for the first good). This case contains all those turned up in Theorems 5.1 and 5.3; generalization along the lines of Theorem 5.2 is left to the reader.[32]

Since g^1 is to reflect the augmentation of the first commodity itself, it is obviously reasonable to assume that $g_1^1 \neq 0$.[33] Actually, we need only assume that x_1 is uniquely determined given b and g^1, i.e. that there exists a function ϕ, such that

$$x_1 = \phi(g^1(x_1, b), b). \tag{5.29}$$

With this assumption, our previous results enable us to handle this relatively general case.

Theorem 5.4. If quality change satisfies (5.28) and (5.29), its effect on the true cost-of-living index can be equivalently represented as a set of price adjustments. The percentage adjustment in the first price depends at most on the amount of the first commodity; the percentage adjustment in the ith price $(i = 2, \ldots, n)$ depends at most on the amount of the first and ith commodities.[34]

Proof. In view of (5.29), every $g^i (i = 2, \ldots, n)$ can be written as a function of g^1, x_i, and b. Thus:

$$g^i(x_1, x_i, b) = h^i(g^1, x_i, b) \quad (i = 2, \ldots, n). \tag{5.30}$$

We shall break up the effect of a change in b into its effects on the various commodities, as follows. Let the b appearing as an argument of g^1 be denoted b_1; let the b appearing as an argument of h^i be denoted b_i $(i = 2, \ldots, n)$. We shall begin with all the b_i equal to unity and shall change them to their common post-quality-change value, denoted $\bar{b}$, one at a time.

Thus, set all the $b_i = 1$, save b_1 and consider the effect of changing b_1 from unity to $\bar{b}$. By (5.30), b_1 enters the utility function only through g^1, and hence the condition of (A) of Theorem 5.1 is satisfied. It follows that the effect of b_1 on y can be equivalently represented as an adjustment in p_1. That adjustment (in percentage terms) depends only on x_1 and not on the other elements of x. Further, in view of Lemma 5.3, that adjustment does not depend on the values of the b_i $(i = 2, \ldots, n)$, so there is no need to remake it when we change those values.

Now move b_2 from unity to $\bar{b}$, keeping $b_1 = \bar{b}$ and $b_i = 1$ $(i = 3, \ldots, n)$. With b_1 fixed, g^1 depends only on x_1, so that h^2 depends only on x_1, x_2, and b_2. It is clear that the condition of (A) of Theorem 5.3 is satisfied, so that the effect of the change in b_2 can be equivalently represented as an adjustment in p_2. That adjustment (in percentage terms) depends at most on x_1 and x_2, and, as before, is independent of the values of the b_i $(i = 3, \ldots, n)$.

Next, move b_3 and adjust p_3. This adjustment is independent of the other b_i $(i = 4, \ldots, n)$ and also independent of b_2. Proceeding in this way, we account for all effects of the quality change and the theorem is proved.

Thus any quality change in the first good, every effect of which can be represented as an augmentation of the services of some good[35] can be handled by adjusting in the cost-of-living index the prices of every good whose services are so augmented and *only* the prices of those goods. In the simplest case of this, given in (5.28), those adjustments (taken in percentage terms) depend at most on the quality of the first good purchased, and possibly on the purchased quantity of the good in question. These price adjustments can be made independently. More complicated cases along the lines of Theorem 5.2 can also be handled. Save in the very simplest of all cases, where only the first good itself is augmented, will a change in the price of the good whose quality has changed be sufficient. (Even then, unless the augmentation is constant, the price change will depend on the quantity of the first good that is purchased.) An adequate treatment of quality change in cost-of-living indices must pay attention to cross-good effects.[36]

FOOTNOTES FOR ESSAY I

[1] An exception is the theory of hedonic price indices where a quality change is regarded as providing a new bundle of old underlying attributes. See Court (1939), Griliches (1961, pp. 173–96), Lancaster (1966), and Stone (1956).

[2] Intertemporal comparisons which do not involve the same set of consumers at both times or geographical comparisons also sharply point up the problem. Following this testament to our ordinalist purity, it is only fair to remark that if the results of our work are to shed light on the construction of a cost-of-living index for a society or even a class within that society the existence of a 'representative consumer' must be assumed. In general, to draw welfare conclusions from aggregate price and quantity data requires interpersonal utility comparisons. For a full discussion of this point, see Samuelson (1947).

[3] Yet this is not inevitable. One can ask how the cost of living in the United Kingdom changed as seen with American tastes or how a man of today would view nineteenth-century price changes.

[4] As already observed, there is a further set of questions in which the tastes are neither those of today nor those of yesterday but are those of a wholly different third situation. For some purposes, these are quite interesting questions to ask, but we shall have nothing to say about them directly in this paper. When the indifference map used in the comparison is one not tied to the situations to be compared, then, of course, we are in the situation envisaged in existing theoretical treatments.

[5] In the case of international or interregional comparisons, both questions have equal interest. The fact that the answers may be quite different is then an inevitable consequence of the fact that people differ. The answer to the question: 'How much income would just make an American with income 100 willing to face British prices?' is not the same as that to the question: 'How much income would make an Englishman indifferent between continuing to face British prices and facing American prices with an income of 100?' Both questions are equally interesting, but they are obviously different. There *is* generally no one answer to both questions and no point in attempting to construct a single index which answers both. One way of looking at the analysis of the next section is as a demonstration of the way in which the answers to the two questions are related if British and American tastes differ in the particular way parametrized in that section.

[6] Note, however, that a policy choice made at the start of the process which did not foresee the taste changes would opt for path B. This is very similar to the myopia problem considered by Strotz (1955–6).

[7] In fact, this proposition is not true if price and income changes are large. This is because of yet another ambiguity in comparing today and yesterday that we have not discussed. The theory of the true cost-of-living index compares the expenditures required yesterday and today to reach a particular indifference curve on a stated indifference

map. But *which* indifference curve is to be used? The natural choices are the indifference curve tangent to yesterday's budget constraint and that tangent to today's. If the indifference map is not homothetic, however, a true cost-of-living index based on the first of these curves (Index A) will not generally coincide with that based on the second (Index B). Yet a moment's consideration reveals that it is Index A which is bounded from above by a Laspeyres index and Index B which is bounded from below by a Paasche index. Unless either the indifference map is homothetic (or obeys other special conditions) or price and income changes are sufficiently small to make Indices A and B close together, there is no reason why the Laspeyres index must lie above Index B or the Paasche index below Index A. Further, both A and B are equally valid and interesting indices.

In this paper, we have, for convenience, concentrated on Index A, that corresponding to the indifference curve tangent to yesterday's budget constraint. Most of our results are equally applicable to Index B, that corresponding to the indifference curve tangent to today's budget constraint. When reading statements about the bounds set by Paasche and Laspeyres indices, however, the discussion of this footnote should be kept in mind. The text implicitly includes the assumption that Index A and Index B do in fact coincide, and we have proceeded on the assumption that in fact the index under discussion is known to be bounded by the Paasche and Laspeyres indices for the case of no taste change. Without that assumption, statements about such bounds apply as statements about the relationship of the bounding index (Laspeyres or Paasche) to the appropriate true cost-of-living index (A or B). We have tried not to overburden the exposition by being explicit about this save in this footnote.

For a discussion of the problems just discussed see v. Hofsten (1952, pp. 28–9) or Malmquist (1953, pp. 221–3).

[8] Note that the implication is not that the true index lies closer to a Paasche index than to a Laspeyres. One does not know this. What one does know is that the Paasche puts a lower bound on changes in the true index, while a Laspeyres fails to have a known relation to it.

The asymmetry between Paasche and Laspeyres indices when tastes change is observed by Malmquist (1953, p. 211).

[9] We treat this case as being the simplest one to analyze. Further, the particular parametrization used not only appears in the theory of technological change but also reappears in the analysis of quality change given below as a result rather than an assumption. Of course, the present section is largely meant as an example of what can be done if an explicit model of taste change is adopted. The necessity for further work is obvious.

[10] Such cases as these may be somewhat more general than the sort of learning effect example given above and continued below. Thus, suppose that the first and second commodities in some sense serve the same needs, so that the utility function can be written as $v(g(bx_1,x_2), x_3, \ldots, x_n)$. Then a change in b might be interpreted as a change in the relative efficiency of the first two commodities in serving those needs, as perceived by the consumer. (Of course, the special form of the utility function in this case has implications for the true cost-of-living index beyond those developed below for the more general case considered in the text.) $u(\cdot)$ serves as a utility function for current *and* base-period tastes. If taste change is solely first good augmenting then the units of b can always be chosen such that the first argument can be written as x_1 in the base period. (Also notice for this section $u(\cdot)$ is a function of n arguments. This notation is inconsistent with that of later sections but no confusion should follow.)

[11] If ψ is a scalar-valued function of the vector w, then $\psi(\cdot)$ is said to be (strictly) quasi-concave if for each scalar ξ the set $\{w : \psi(w) \geqslant \xi\}$ is (strictly) convex. See Arrow and Enthoven (1961).

[12] Equation (3.10) is an instance of the class of envelope theorems frequently encountered in the constrained minimization (and maximization) problems of economics. For a discussion of envelope theorems, see Samuelson (1947, pp. 34–5).

[13] Note that there are two effects. An increase in b makes it cheaper today to attain a given utility level, but it also raises the utility level which would have been achieved with yesterday's income and prices. If we were analyzing quality change rather than taste change, only the former effect would be present.

[14] More complicated theorems can be derived from Theorem 3.1, Corollary 3.1, and Lemma 3.6 (or from Theorem 3.2 using the chain property of the true cost-of-living index). For example, we know that if demand for the first good is price elastic and its price has risen ($p_1 > \hat{p}_1$) and if we know that prices have risen for all goods that are gross complements for the first and have fallen for all goods that are gross substitutes for the first good, then we know that $(\partial y / \partial b) > 0$. (Assuming, of course, that for the relevant values of prices, all goods other than the first remain either gross complements or gross substitutes for the first good. The assumption that the sign of η_{j1}, $j = 2, \ldots, n$, or of $(\eta_{11} + 1)$ does not change when prices change is implicit in much of the discussion that follows.)

[15] It may be thought that these results are obvious. It is natural to expect, for example, that in the situation being analyzed substitutes for the first good will decline in importance and complements will increase. While it is clear that one should indeed expect this as part of the intuitive

meaning of 'substitutes', however, it is not at all clear to us that one would automatically apply such intuition to *gross* substitutes rather than to *net* substitutes or to substitutes defined in yet some different way.

[16] As before, it is implicitly assumed that we remain in ranges of prices in which the elasticity stays on the same side of minus unity and substitute-complement relationships are not reversed.

[17] If the first good is not inferior, certain cases are ruled out. Thus, in this case, the jth good must be a net substitute for the first good if it is also a gross substitute. Similarly, if the demand for the first good is inelastic, η^n_{11} must be greater than -1.

[18] The systems (4.1)–(4.4), (4.9), and (4.5)–(4.8) are the well-known conditions of Kuhn-Tucker-Lagrange (KTL). The assumption of nonsatiation of consumption guarantees that if (4.1)–(4.9) is solved for y then $(y/\hat{y})$ is the true cost-of-living index. The proof of the optimality of KTL for quasi-concave programming problems with nonsatiation is given in Arrow and Enthoven (1961, pp. 783–8). Nonsatiation also implies that the equilibrium values of λ and $\hat{\lambda}$ are positive.

[19] Arrow (1958, p. 85) discusses the use of demand reservation prices in the construction of a cost-of-living index.

[20] We have already discussed the problem of deciding whether to treat a given change as one in quality or one in tastes. There is a less basic decision as to whether a change in quality should be treated as such or as the appearance of a new good and the disappearance of an old one. This decision (unlike the former one) is largely a matter of convenience. In this section, we assume that it has been made in favor of retaining the same name (or subscript) for a good before and after the change, i.e. in favor of treating the change as one in the quality of a given good.

[21] The use of hedonic price indices (see the references in note 1) is the most sophisticated way now known of using such market information to obtain price adjustments for quality change. It should come as no surprise that the extension of the results of hedonic price index investigations outside the sample period in which the market observations are made is strictly appropriate only in the repackaging case. The theory of hedonic price indices treats a new quality of a given good as a repackaging of a bundle of underlying attributes. Only if the attributes enter the utility function through the 'package' rather than directly, will hedonic price index adjustments be more than locally appropriate. Obviously, to say this is not to disparage the usefulness of hedonic price indices in practice.

[22] For many purposes it is simpler to regard refrigerator services as entering the utility function directly than it is to leave them out. Consumer theory deals with goods traded in the market place, not with later composites of them made up by consumers (such as home-refrigerated ice cream). In any case, to say that refrigerators enter directly rather than

through other goods is a matter of notation at the level of abstraction of most treatments of consumer theory.

[23] There is no reason other than one of convenience why b has to be a scalar. Quality change may take place in more than one attribute of the first good, in which case b would be replaced by $(b_1, \ldots, b_k)$ and the analysis would be essentially unchanged.

[24] It may be noted that the present problem differs from that of taste changes discussed in section III above in that the change in the utility function is 'embodied' in the first good rather than being 'disembodied'. The parallel to models of embodied and disembodied technical change in production functions is obvious, extending the well-known parallel between the theory of the utility-maximizing consumer and the theory of the cost-minimizing firm. Indeed, some of the results of this section also parallel some of the results in the analysis of such models. We shall return to this in a later footnote.

[25] We assume that $u(\cdot)$ is a strictly quasi-concave function of its first n (nonnegative) arguments and restrict our attention to interior minima.

[26] Note that the result is just that which would be obtained ignoring the effects of b on x. Thus a small unit increase in b raises u by u_b which allows a decrease in expenditure by u_b/λ, since $1/\lambda$ is the marginal cost of a unit of utility. As in the analogous case in section III (and as in the lemma which follows), this is an envelope theorem.

[27] If the new and old varieties of the first good coexist on the same (perfect) market, however, their relative prices will code all the information needed for local adjustment. See the discussion above.

[28] It is natural to take $g(x_1,1) = x_1$, i.e. $g^*(x_1,1) = 1$, but this is not required for our results.

[29] The situation is very similar to that in models of embodied technical change in which a capital aggregate is to be formed or the effect of technical change removed by the use of a quality-corrected capital index, that is, by adjusting the prices of capital goods of different vintages. Under constant returns, technical change must be capital augmenting, analogous to part (B) of the theorem. Under a generalized form of constant returns in which the production functions are homogeneous of degree one in labor and some function of capital, technical change must be capital-altering, a kind of change analogous to the variable repackaging of part (A) of the theorem. See Fisher (1965).

[30] It is natural to take $g(x_1, \ldots, x_m, 1) = x_1$, i.e. $g^*(x_1, \ldots, x_m, 1) = 1$, but this is not required for our results.

[31] It is natural to take $g(x_1,x_2,1) = x_2 = g(0,x_2,b)$, i.e. $g^*(x_1,x_2,1) = 1 = g^*(0,x_2,1)$, but this is not required for our results.

[32] It is natural to take $g^1(x_1,1) = x_1$ and $g^i(x_1,x_i,1) = x_i = g^i(0,x_i,b)$

($i = 2, \ldots, n$), i.e. $g^{*1}(x_1,1) = 1$ and $g^{*i}(x_1,x_i,1) = 1 = g^{*i}(0,x_i,b)$ ($i = 2, \ldots, n$), but this is not required for our results.

[33] If $g^1_1 = 0$ in some open neighborhood in the $x_1 - b$ plane in which $g^1_b \neq 0$, then b enters the utility function in that neighborhood in some way other than by augmenting the services of the commodities.

[34] If g^1 takes the form of (B) of Theorem 5.3, only dependence on the first commodity is involved; if g^1 takes the form of (B) of Theorem 5.1, the percentage adjustment in p_1 is a constant.

[35] This is quite general in the small, but not in the large.

[36] Is it really much more difficult to say, for example, how the introduction of larger, more powerful cars affects the enjoyment of the services of other prestige items than it is to say how such introduction affects the enjoyment of the services of cars? Both evaluations seem hard to make, but the second one is made in practice. Admittedly, however, the second evaluation can be made implicitly through the use of market data if new and 'old' (but not necessarily used) cars sell on the same perfect market. Even then, as we have seen, that adjustment will generally only suffice while that market situation lasts.

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