The Theory and Application of Differential Games

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[illegible] Differential Games

THE THEORY OF HAMILTONIAN DYNAMICAL SYSTEMS, AND AN APPLICATION TO ECONOMICS

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INTRODUCTION

A Hamiltonian dynamical system (HDS) naturally arises in the standard control problem involving optimization over time. On this ground alone, a systematic study of the basic structure of the general HDS should be extremely useful for mathematical control theory. Applications of HDS theory extend beyond models involving optimization. The classic studies of such systems were motivated by problems in celestial mechanics. While much of the analysis of my lecture will be motivated by the normative (optimizing) model of macroeconomic growth, I will show in passing how HDS theory may be useful in analyzing many positive ("nonoptimizing") models of macroeconomic growth.

The basic approach of this lecture is to relate stability properties of the HDS to the geometry of the underlying Hamiltonian function (HF) generating that HDS. For simplicity, the present analysis is restricted to continuous-time models, although the Hamiltonian approach is equally powerful in analyzing discrete-time models.

The emphasis of this lecture on the geometry of the HF as an important determinant of (global) stability of the HDS is drawn directly from a joint paper with my colleague David Cass [4], which will appear in the Journal of Economic Theory. All of the basic propositions -- i.e., those relating global stability to Hamiltonian steepness -- are from the Cass-Shell article.

LOCAL ANALYSIS FOR THE TIME-AUTONOMOUS HAMILTONIAN FUNCTION: THE CLASSICAL THEOREM OF POINCARE

Let $\xi(t)$ and $\eta(t)$ be m-dimensional vectors dependent upon time t. The HF $H(\xi(t), \eta(t))$ is said to be time-autonomous, since H depends on t solely through ξ and η. $H(\xi,\eta)$ is said to generate the HDS

$$\dot{\xi} = -\frac{\partial H}{\partial \eta}, \quad \dot{\eta} = \frac{\partial H}{\partial \xi}, \tag{1}$$

where here $\partial H/\partial \xi$ and $\partial H/\partial \eta$ are vectors of partial derivatives (later, vectors of generalized gradients) and $\dot{\eta}$ and $\dot{\xi}$ are vectors of time derivatives. I suppress the dependence on t whenever confusion will not arise. Assume that the above HDS possesses a rest point, which can be chosen as the origin without loss of generality. Then

$$\frac{\partial H(0,0)}{\partial \eta} = 0 = \frac{\partial H(0,0)}{\partial \xi}. \tag{2}$$

I follow the approach of classical mechanics; see, e.g. [10], especially pp. 76-81. Make a linear approximation to the HDS about the rest point. If λ is a characteristic root of the associated linear system, then it must satisfy the characteristic equation

$$\begin{vmatrix} H_{\eta\xi} + \lambda I & H_{\eta\eta} \\ H_{\xi\xi} & H_{\xi\eta} - \lambda I \end{vmatrix} = 0, \tag{3}$$

where $H_{\xi\xi}$, $H_{\eta\eta}$, $H_{\xi\eta}$ and $H_{\eta\xi}$ are m × m matrices of cross partials evaluated at (0,0) and I is the m × m identity matrix. I assume that $H(\cdot)$ is twice continuously differentiable. Notice the abundant symmetry in the characteristic equation (3) for the linearized HDS. $H_{\eta\xi} = H'_{\xi\eta}$, $H_{\eta\eta} = H'_{\eta\eta}$, and $H_{\xi\xi} = H'_{\xi\xi}$, where primes denote matrix transposition. Substituting in (3), transposing the determinant on the left, and interchanging rows and columns yields

$$\begin{vmatrix} H_{\eta\xi} - \lambda I & H_{\eta\eta} \\ H_{\xi\xi} & H_{\xi\eta} + \lambda I \end{vmatrix} = 0. \tag{4}$$

Comparing (3) and (4) gives Poincaré's result: If λ is a root to the characteristic equation for the linearized HDS, then $-\lambda$ is also a root.

This Poincaré theorem is at once extremely simple and also suggestive of deep fundamental results. If we could rule out roots with zero real parts, we would then have established a saddlepoint result for the linearized HDS in 2m-dimensional (ξ, η) phase space: In the neighborhood of the rest point (0,0), the manifold of (forward) solutions tending to the rest point as $t \to +\infty$ is of dimension m, and the manifold of (backward) solutions tending to the rest point as $t \to -\infty$ is also of dimension m.

Since zero real parts would seem to be something of a mathematical accident, it would seem apparent that <u>the local saddlepoint property is generic</u> in the sense of global analysis. That is, HDS's possessing this property are dense in the class of all HDS's possessing a rest point.

I know of no formal proof that the saddlepoint property is generic. There are even some reasons that might make one skeptical of the validity of saddlepoint genericity: (1) Since the λ's are solutions to a polynomial equation, "outcomes" are further constrained; e.g., if λ is a complex root, then its complex conjugate is also a root. (2) Somehow, real-life planetary motion seems to be able to sustain itself.

Another approach suggested by examples from economics (see [17,18,19]) is to seek sufficient conditions in terms of the geometry of the HF that ensure the saddlepoint property. The Poincaré result requires little in the way of restrictions on the HF, viz., the existence of a rest point and twice continuous differentiability. (It should be noted that somewhat stronger conditions may be needed for the saddlepoint property to carry over from the linearized version to the HDS itself.)

Rockafellar [12], in an important mathematical paper stimulated by problems in economics (cf., [17]), establishes -- among other things -- a global version of the saddlepoint property for the case in which the rest point (0,0) is a saddlepoint of the HF $H(\xi,\eta)$ with $H(\cdot)$ <u>strictly</u> convex in ξ and <u>strictly</u> concave in η. Therefore, if the HF is strictly convex-concave and possesses a saddlepoint (0,0), then the HDS possesses a saddlepoint (0,0).

In optimal economic growth, the HF is interpreted as the maximized value of socially imputed net national product with ξ as output prices and η as input stocks. It is thus natural to assume at least convexity in prices ξ and concavity in stocks η.

Moreover -- although various schools of economics seem to "rediscover" this in particular examples -- it is quite natural for a competitive dynamical system to possess this saddlepoint property in phase space. Such intertemporal development is shown in the phase-plane of Figure 1, where $m = 1$.

From Figure 1, we see that for each initial endowment $\eta(0)$ there exists at most one (locally, exactly one) price $\xi(0)$ for which $(\xi(t),\eta(t)) \to (\xi^*,\eta^*)$, the rest point, as $t \to \infty$. Errant paths -- those not tending to (ξ^*,η^*) -- seem to violate additional conditions for optimality, such as nonnegativity of prices or asset-market clearing, or some transversality condition, or bounded value of capital. Such arguments have, however, only been firmed-up for special cases (see, e.g., [5,17,19]) and seem to rely for proof on topological properties appropriate to the phase-plane but inappropriate for higher dimensional phase spaces.

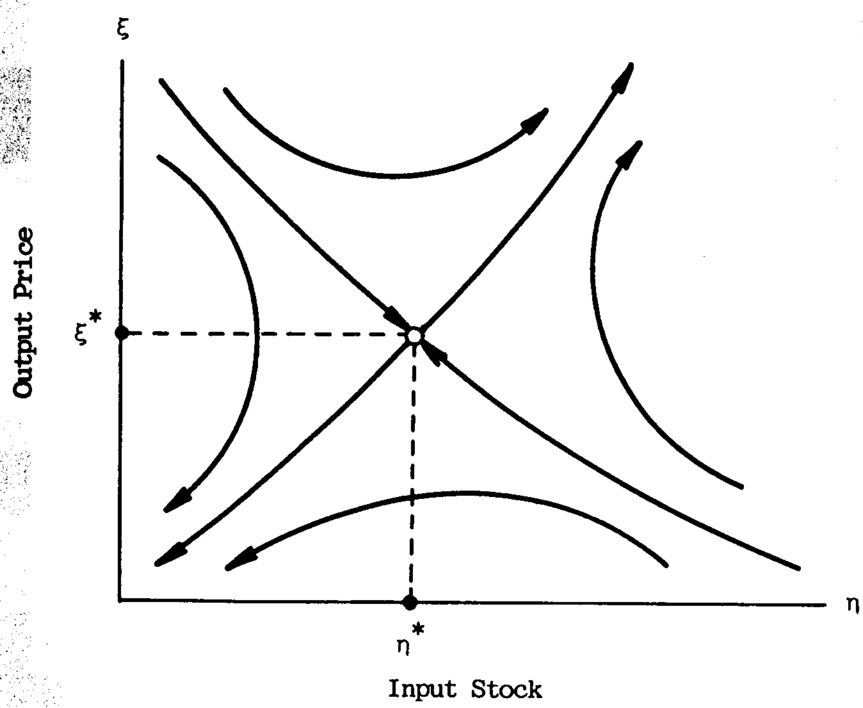

FIGURE 1

In what follows, I shall report on the Cass-Shell approach to Hamiltonian dynamics. We start with the geometry of the Hamiltonian function, but rely on global steepness properties rather than measures of convexity and concavity. We also jump over the step of studying the dimensionality of the stable manifold. Instead, we add nonnegativity and transversality conditions to our definition of the HDS and investigate the global stability of the rest point (ξ^*, η^*), or more naturally the global stability of the "real" (nonprice) component, η^*. Furthermore, we are able to accommodate an important class of non-time-autonomous HDS. Before turning to the stability analysis, I must generalize our notion of the HF and the HDS it generates.

THE HAMILTONIAN FUNCTION: AN ECONOMIC REPRESENTATION

Not only does a Hamiltonian function generate a Hamiltonian dynamical system which is convenient for dynamic economic analysis, but the HF is also an interesting construct for static economic analysis. Changing notation from the previous sections, let $k(t)$ be an m-vector of capital stocks at time t, let $z(t)$ be an m-vector of net investment goods output at time t so that $\dot{k}(t) = z(t)$, let $q(t)$ be an m-vector of present prices of investment goods at time t, let $c(t)$ be (scalar) consumption or "instantaneous social utility" at time t, and let $p(t)$ be the (scalar) present price of consumption at time t. Given $(p,q) \geq 0$, the present value of net national output is derived by maximizing $pc + qz$ subject to technology and endowments of capital k, and endowment of the sole (for convenience) fixed factor, ℓ, (interpreted as labor). If all capital stock inputs are feasible, this static optimization yields an HF (interpreted as the present value net national output)

$$H(p,q,k,\ell)$$

defined over the nonnegative orthant $\{(p,q,k,\ell) : (p,q,k,\ell) \geq 0\}$. For convex technologies with free disposal, this convenient Hamiltonian representation of technology is fully equivalent (and often easier to work with) than the usual input-output set-theoretic representation or the related production-frontier representation. The HF has the following properties:

(a) H is nondecreasing in p and ℓ;

(b) H is linear homogeneous in (p,q). Without loss in generality, since convex technologies can always be described as constant-returns technologies after introducing a fictitious commodity, we also assume that H is linear homogeneous in (k,ℓ);

(c) H is convex in (p,q) and concave in (k,ℓ);

(d) The generalized gradients satisfy

(i) $\frac{\partial H}{\partial(p,q)} = (c,z)$ and

(ii) at least for $(p,q) \geq 0$,

$\frac{\partial H}{\partial(k,\ell)} = (r,w)$, where by duality, r is the vector of competitive rental rates and w is the wage rate on labor.

For further discussion of the static aspects of the above HF, $H(p,q,k,\ell)$, see Cass [3], Lau [7], and Cass-Shell [4].

GENERALIZED HAMILTONIAN LAWS OF MOTION

In terms of the HF, $H(p(t),q(t),k(t),\ell(t))$, our generalized HDS becomes

$$\left.\begin{aligned} \dot{k}(t) &= \frac{\partial H(p(t),q(t),k(t),\ell(t))}{\partial q(t)}, \\ \dot{q}(t) &= -\frac{\partial H(p(t),q(t),k(t),\ell(t))}{\partial k(t)}. \end{aligned}\right\} \tag{5}$$

For steady state analysis to apply, $\ell(t)$ must grow at an asymptotically constant rate. Without further loss of generality, I can set $\ell(t) \equiv 1$. Of course, initial capital stock endowments are exogenously specified so that $k(0) = k_0$. If $(p(t),q(t)) \geq 0$, then (5) is consistent with the usual perfect-foresight competitive asset-market clearing equations from economics.

OPTIMAL ECONOMIC GROWTH WITH POSITIVE DISCOUNTING

As an economic example, I propose to investigate the problem of maximizing social welfare as given by

$$\int_0^\infty c(t)e^{-\rho t}\,dt, \tag{6}$$

where $\rho > 0$ is a constant scalar. (I could include the limiting case $\rho = 0$, but although analytically simpler, it would open up various time-consuming caveats about boundedness of the criterion functional and transversality, etc. Instead see [4]. Maximization is subject to technological constraints, initial endowments, k_0, and labor availability, $\ell(t) \equiv 1$. To my knowledge, for all cases in which the Maximum Principle has been worked out, HDS(5) is necessary for optimality. Similarly, I take as a necessary condition that

$$\dot{p}(t)/p(t) = -\rho$$

or without loss in generality that

$$p(t) = e^{-\rho t} \tag{7}$$

I must also add a transversality condition:

$$\lim_{t \to \infty} q(t)k(t) = 0. \tag{8}$$

In discrete-time Weitzman [20] has shown that (8) is necessary for optimality. I suspect that techniques similar to those developed in [4] should be useful in establishing the necessity of (8) in continuous time, but Cass and I have not verified this conjecture.

It will be convenient to switch from present prices to current prices. Define $Q \equiv q/p$ and $H(Q,k) \equiv H(1,Q,k,1) \equiv H(1,q/p,k,1)$ The system (5),(7), and (8) can be written as

$$\left.\begin{aligned} &\dot{k} = \frac{\partial H(Q,k)}{\partial Q}, \ k(0) = k_0, \\ &\dot{Q} = -\frac{\partial H(Q,k)}{\partial k} + \rho Q, \ Q(0) \geqq 0, \\ &\lim_{t \to \infty} Qe^{-\rho t}k(t) = 0. \end{aligned}\right\} \tag{9}$$

GLOBAL STABILITY OF OPTIMAL ECONOMIC GROWTH

I assume here that there exists a rest point, (Q^*,k^*), to HDS(9). For a proof of existence, see [4].

I have promised to provide geometrical conditions on the HF, $H(Q,k)$, which will ensure global stability of the steady state capital stock, i.e., that ensure $\lim_{t\to\infty} k(t) = k^*$. The stability condition, which is due to Cass-Shell [4], is

$$\left.\begin{array}{l}\underline{\text{Stability Assumption}}\text{: For every } \varepsilon > 0\text{, there is a } \delta > 0 \\ \text{such that } \|k-k^*\| > \varepsilon \Longrightarrow \\ (Q-Q^*)\dfrac{\partial H(Q,k)}{\partial Q} - \dfrac{\partial[H(Q,k)-\rho Q^*k]}{\partial k}(k-k^*) > \delta - \rho(Q-Q^*)(k-k^*).\end{array}\right\} \quad (S)$$

(S) is the basic stability assumption, although for economic problems attention is further restricted to HF's which are in addition convex in Q and concave in k, so that

$$(Q-Q^*)\frac{\partial H(Q,k)}{\partial Q} - \frac{\partial[H(Q,k)-\rho Q^*k]}{\partial k}(k-k^*) \geqq 0. \tag{10}$$

Even if the ε-δ bounds in S are removed, the weakened condition still ensures uniqueness of k^*. If (S) is strengthened in the obvious way, it will provide for the uniqueness and global stability of the full rest point, (Q^*,k^*).

(S) is a "steepness" requirement on the HF, looking from any point, (Q,k) with $k \neq k^*$ to (Q^*,k^*), steepness must be bounded above that given by the quadratic form $-\rho(Q-Q^*)(k-k^*)$.

Cass-Shell [4] provides more motivation of condition (S) -- both from the economic and geometric points of view. Here, I merely remark that (S) can be thought of as a generalization of the bounded-value-loss condition of Radner [11] and reduces to Radner's condition when $\rho = 0$.

Condition (S) immediately suggests the choice of Lyapunov valuation function for stability analysis:

$$V = (Q-Q^*)(k-k^*). \tag{11}$$

Time differentiation in (11) and application of (10) yields:

$$\left.\begin{aligned} d(Ve^{-\rho t})/dt &\sim \dot{V} - \rho V \\ &= (Q-Q^*)\frac{\partial H(Q,k)}{\partial Q} - \frac{\partial[H(Q,k)-\rho Qk]}{\partial k}(k-k^*) \\ &\qquad -\rho(Q-Q^*)(k-k^*) \\ &= (Q-Q^*)\frac{\partial H(Q,k)}{\partial Q} - \frac{\partial[H(Q,k)-\rho Q^*k]}{\partial k}(k-k^*) \\ &\geqq 0. \end{aligned}\right\} \tag{12}$$

The transversality condition (9) requires that

$$\lim_{t\to\infty} Ve^{-\rho t} \leqq \lim_{t\to\infty} [Qe^{-\rho t}k + Q^*e^{-\rho t}k^*] = 0 \tag{13}$$

and hence, $V \leq 0$ and $\lim_{t\to\infty} V = V^* \leq 0$.

<u>Proof of global stability</u>. Suppose that $\lim_{t\to\infty} k(t) = k^*$ were not true. Then for some $\varepsilon > 0$ there would be a sequence of points $\{t_j\}$ such that $\|k(t_j)-k^*\| > 2\varepsilon$. Then if we assume uniform continuity[1] of $k(t)$ on the halfline $[0,\infty)$ -- there would also have to be a sequence of intervals $\{[\underline{t}_j,\bar{t}_j]\}$ such that $\bar{t}_j-\underline{t}_j > \Delta t > 0$ and $\|k-k^*\| > \varepsilon$ for $t \in [\underline{t}_j,\bar{t}_j]$. From (12), it follows that

$$\dot{V} = (Q-Q^*)\frac{\partial H(Q,k)}{\partial Q} - \frac{\partial[H(Q,k)-\rho Q^*k]}{\partial k}(k-k^*) + \rho V$$

so that (S) implies there is a $\delta > 0$ such that

$$\dot{V}(t) > \delta \text{ for } t \in [\underline{t}_j,\bar{t}_j].$$

Hence, for t' sufficiently large, it follows that both

$$V^* - \delta \leq V(t) \leq V^*$$

and

$$V(t) \geq V(t') + \sum_{t' \leqq \underline{t}_j \leqq \bar{t}_j \leqq t} \delta(\bar{t}_j-\underline{t}_j) \geq V(t')$$

$$+ [\max\{j : t' \leq \underline{t}_j \leq \bar{t}_j \leq t\} - \min\{j : t' \leq \underline{t}_j \leq \bar{t}_j \leq t\}]\delta\Delta t$$

for $t \geqq t'$, which are inconsistent, establishing that a solution to (9), if it exists, must exhibit stability in the sense that $\lim_{t\to\infty} k(t) = k^*$.

NOTE

[1]Uniform continuity is established in [4].

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