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Source: *Journal of Political Economy*, Vol. 111, No. 1 (February 2003), pp. 103-123
Published by: The University of Chicago Press
Stable URL: http://www.jstor.org/stable/10.1086/344803
Accessed: 29/08/2015 16:16

Equilibrium Bank Runs

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We analyze a banking system in which the class of feasible deposit contracts, or mechanisms, is broad. The mechanisms must satisfy a sequential service constraint, but partial or full suspension of convertibility is allowed. Consumers must be willing to deposit, ex ante. We show, by examples, that under the so-called "optimal contract," the postdeposit game can have a run equilibrium. Given a propensity to run, triggered by sunspots, the optimal contract for the full predeposit game can be consistent with runs that occur with positive probability. Thus the Diamond-Dybvig framework can explain bank runs as emerging in equilibrium under the optimal deposit contract.

I. Introduction

The theoretical literature on bank runs is based on the early work of Bryant (1980) and the now-classic model of Diamond and Dybvig (1983). When simple deposit contracts are used, the contract supporting the efficient allocation is shown to support a bank-run equilibrium as well. However, when convertibility is suspended, the bank-run equilibrium is eliminated. The current state of the art is work by Green and Lin (2000*a*, 2000*b*), inspired in part by Wallace (1988, 1990). Green

We thank Patrick Bolton, John Duggan, Huberto Ennis, Ed Green, Peter Howitt, Todd Keister, Bob Lucas, Bruce Smith, Neil Wallace, two referees, and seminar participants at the Cleveland Fed, Cornell, the Cornell/Pennsylvania State University Macro Workshop, Indiana University/Purdue University at Indianapolis, Miami, New York University, Princeton, and Texas for helpful discussions and comments on an earlier, much different version.

[*Journal of Political Economy,* 2003, vol. 111, no. 1]

and Lin allow for a broad class of banking contracts. Because there is aggregate uncertainty, the sequential service constraint precludes achieving the full-information first-best. They show that the mechanism that supports the constrained-efficient allocation does not permit bank-run equilibria. Thus the literature that started with Diamond-Dybvig is unable to explain bank runs until now. Bank runs are historical facts. If bank runs were impossible, then much of banking policy would be directed toward a nonissue. Our goal is to put "runs" back in the bank-runs literature. In particular, we investigate the possibility of equilibrium runs on banks that can write sophisticated contracts in which the current withdrawal depends on the history of withdrawals. We provide the first examples in the Diamond-Dybvig literature in which a bank run can occur in equilibrium *under the optimal deposit contract within a broad class of mechanisms that includes suspension schemes.*[1] We show by examples that, for some parameters, the mechanism that supports the constrained-efficient allocation for the postdeposit game also permits a run equilibrium. The nonrun equilibrium to the postdeposit game is also an equilibrium to the predeposit game. The run equilibrium is not, because consumers would not make deposits if they knew that a bank run would follow. If bank runs are triggered by sunspots, then the optimal contract to the predeposit game can have a run equilibrium if the propensity to run is small. For greater propensities to run, the optimal contract to the predeposit game is immune to runs, but the welfare of the constrained-efficient allocation may not be achievable.

The intuition for our results is that the "optimal contract" maximizes welfare subject to an incentive compatibility constraint, which requires a patient consumer to weakly prefer choosing period 2 to period 1. This incentive compatibility constraint presupposes that the other patient consumers choose period 2. If, instead, the other patient consumers are believed to choose period 1, it is possible that our patient consumer would prefer to choose period 1, in which case there is a run equilibrium to the postdeposit game.

There are two important distinctions between our model and the model of Green and Lin (2000*a*, 2000*b*) that explain the differences in the results: (1) We allow the utility functions, of period 1 consumption for the impatient and of period 2 consumption for the patient, to differ across the two types; Green and Lin do not. Thus we allow the incentive compatibility constraint to bind at the optimal contract. (2) Green and Lin assume that the consumer knows the *clock time* at which she arrives at the bank, which tells her roughly her position in the queue. Knowing

[1] Diamond and Rajan (2001) develop a model in which the possibility of a bank run affects bankers' bargaining power in renegotiating loan contracts with borrowers. If a run occurs, depositors capture the loans and renegotiate with borrowers directly. However, it is the *threat* of a run that disciplines bankers, and a run cannot occur in equilibrium.

the time is crucial to their backward induction argument. We have no clock. We assume that the consumer decides whether to withdraw or not, knowing her type (patient or impatient) but not knowing her place in line. Once made, her withdrawal decision is irrevocable.

There is a difference between Green and Lin's model and ours that is not crucial. Green and Lin consider direct revelation mechanisms in which all consumers sequentially report their types to the bank as they arrive in period 1. For example, when someone reports "patient," the bank can give her consumption in period 2 but use the information to affect the period 1 consumption of consumers arriving later in the queue and reporting "impatient." In our model, consumers with no intention of withdrawing money in period 1 do not contact the bank. We think of the mere arrival at the queue as essentially a report of "impatient." It is hard to imagine people visiting their bank for the purpose of telling them that they are not interested in making any transactions at the present time. In our indirect mechanisms, a consumer's strategy is simply a choice of which period to visit the bank. However, in Appendix B we adapt our basic model to incorporate complete reporting à la Green and Lin. We show by example that, for some parameters, the direct-revelation mechanism supporting the (new) constrained-efficient allocation also permits a run equilibrium.

In Section II, we describe the postdeposit game. In Section III, we present a two-consumer example in which the constrained-efficient allocation is supported by a contract that also allows for a run equilibrium. In Section IV, we assume that consumers observe a sunspot variable after making a deposit but before choosing when to visit the bank. We show that there can be an equilibrium to the full predeposit game, based on the contract that supports the constrained-efficient allocation in the postdeposit game, that entails a positive probability of runs. For the example of Section III, we calculate in Section IV the highest probability of a bank run such that the bank cannot improve welfare by changing the mechanism to eliminate equilibrium runs. Above this critical probability, the optimal contract to the predeposit game does not permit a run equilibrium; however, welfare is lower than under the constrained-efficient allocation. In Section V, we make our concluding remarks. In Appendix A, we show that our basic result about the postdeposit game, namely that the contract that supports the constrained-efficient allocation also permits a run equilibrium, extends to the case with many consumers and correlated types. In Appendix B, we show that our basic result is robust to allowing direct revelation mechanisms à la Green and Lin, where patient as well as impatient consumers contact the bank in period 1.

II. The Model

There are three periods and a finite number of consumers (the potential bank depositors), N. In period 0, each consumer is endowed with y units of the consumption good. Let α denote the number of *impatient* consumers: each of them derives utility only from consumption in period 1. The remaining consumers are *patient:* each of them derives utility from consumption in period 2. Patient consumers can costlessly store consumption across periods. Let c^1 denote consumption *received in* period 1 and let c^2 denote consumption *received in* period 2. Impatient and patient consumers, respectively, have the utility functions $u(c^1)$ and $v(c^1 + c^2)$. We assume that u and v are strictly increasing, strictly concave, and twice continuously differentiable and that the coefficients of relative risk aversion are less than minus one, or

$$\frac{xu''(x)}{u'(x)} < -1, \quad \frac{xv''(x)}{v'(x)} < -1, \tag{1}$$

for each positive x. Impatient and patient consumers can have different utility functions, motivated by time preference or the interpretation that impatient consumers face extraordinary consumption opportunities.

Let $f(\alpha)$ denote the probability that the number of impatient consumers is α, for $\alpha = 0, 1, \ldots, N$. A consumer's type, impatient or patient, is her private information. In keeping with our assumption that consumers are identical, ex ante, assume that, conditional on a consumer's being patient, the probability that the number of impatient consumers is α, denoted by $f_p(\alpha)$, is the same for all consumers. Using Bayes' rule, we can calculate this as

$$f_p(\alpha) = \frac{[1 - (\alpha/N)]f(\alpha)}{\sum_{\alpha'=0}^{N-1} [1 - (\alpha'/N)]f(\alpha')}$$

for $\alpha = 0, 1, \ldots, N$. We have the following process in mind. First, nature chooses α according to f. Then nature randomly chooses the set of impatient consumers so that, conditional on α, each consumer is equally likely to be impatient. Notice that this overall process allows for correlation among types but also admits the independent and identically distributed (i.i.d.) case.

The investment technology is described as follows. Investing one unit of period 0 consumption yields $R > 1$ units if held until period 2 and yields one unit if harvested in period 1. So far, the only main departure from the Diamond-Dybvig model is that the utility function can depend on a consumer's type.[2]

Following the literature, we focus for the moment on the postdeposit

[2] See also Jacklin (1987) for an extension in this direction.

game, mindful of the fact that if a bank run is anticipated to occur with probability one, no consumer would be willing to deposit in period 0. In Section V, we introduce sunspots and analyze equilibrium runs in the full (predeposit) game.[3] In period 0, the bank designs a deposit contract, which we call the banking mechanism. We assume that the bank seeks to maximize the ex ante expected utility of consumers. To the extent that banking is perfectly competitive, any bank attracting depositors must act in this manner.

Here is the timing of the postdeposit game, which starts after the banking mechanism is in place and consumers have deposited their endowments. At the beginning of period 1, each consumer learns her type and decides whether to arrive at the bank in period 1 or period 2. We require that the mechanism satisfy the following *sequential service* constraint. Consumers who choose period 1 are assumed to arrive in random order. Let z_j denote the position of consumer j in the queue. Because of the sequential service constraint, consumption in period 1 must be allocated to consumers as they arrive at the head of the queue, as a function of the history of transactions up until that point. We further assume that consumer j's withdrawal can be a function only of her position, z_j. That is, we consider indirect mechanisms in which a consumer's strategy is a choice of round, as a function of her type. Arrival in period 1 can be interpreted as a report that the consumer is impatient, but no explicit reports are made.

For $z = 1, \ldots, N$, let $c^1(z)$ denote the period 1 withdrawal of consumption by the consumer in arrival position z. Since optimal mechanisms induce the patient consumers to choose period 2 and since giving period 1 withdrawers more consumption in period 2 hurts incentive compatibility for the patient (and does not help the impatient), remaining resources in period 2 are allocated to consumers who choose period 2. Because of the concavity of v, we can restrict attention to mechanisms that smooth second-period consumption. Therefore, let $c^2(\alpha_1)$ denote the second-period consumption to those who choose to wait until period 2, when the number of consumers choosing period 1 is α_1, for $\alpha_1 = 0, \ldots, N-1$. (Under "truth telling," the impatient and only the impatient choose period 1, and we would then have $\alpha_1 = \alpha$.) The resource condition can be written as

$$c^2(\alpha_1) = \frac{[Ny - \sum_{z=1}^{\alpha_1} c^1(z)]R}{N - \alpha_1}, \quad c^1(N) = Ny - \sum_{z=1}^{N-1} c^1(z). \tag{2}$$

[3] Diamond and Dybvig (1983) are aware of this point and mention the possibility that sunspots could allow runs to occur with small probability, thereby maintaining the incentive to deposit. See also Cooper and Ross (1998).

Thus the banking mechanism, **m**, could be described by the vector

$$\mathbf{m} = (c^1(1), \ldots, c^1(z), \ldots, c^1(N), c^2(0), \ldots, c^2(N-1)),$$

with the interpretation given above. Notice that **m** satisfies the sequential service constraint because the zth consumer to arrive in period 1 receives consumption that depends solely on her place in line. In particular, $c^1(z)$ does not require information about people behind her in line. Let the set of banking mechanisms be denoted by M. Then we have

$$M = \{\mathbf{m} \in \Re_+^{2N}:\ (2) \text{ holds for } \alpha = 0, \ldots, N-1\}.$$

Our set of deposit contracts or mechanisms is fairly broad and allows for partial or full "suspension of convertibility." However, our class of indirect mechanisms is different from the direct mechanisms considered by Green and Lin (2000*a*, 2000*b*). In their model, both patient and impatient consumers arrive at the bank in period 1, at random times, and report their type. In addition to resource constraints and incentive compatibility, their mechanism must also satisfy sequential service. It is consistent with their sequential service constraint to let c^1 depend on how many patient as well as impatient reports have been made earlier. We focus on indirect mechanisms, in which consumers choose when to arrive but do not make explicit reports. Another difference between our model and that of Green and Lin is that they assume the existence of a "clock," so that consumers roughly know their place in line. The clock allows Green and Lin to rule out equilibrium bank runs, by iterated elimination of strictly dominated strategies. Intuitively, a patient consumer who arrives with one second left on the clock will be last in line with high probability. In that case, she prefers to report her type truthfully (i.e., receive consumption in period 2) since whatever has not been harvested will yield the higher return, R. Then a patient consumer who arrives with two seconds left on the clock will know that later-arriving consumers report truthfully, which they show implies that this consumer should report truthfully. Then a patient consumer who arrives with three seconds left on the clock will report truthfully, and so on. We assume that consumers choose when to arrive knowing only whether they are patient or impatient, with no clock.

Definition 1. Given a mechanism $\mathbf{m} \in M$, the postdeposit game is said to have a *run equilibrium* if there is a Bayes-Nash equilibrium in which all consumers choose to withdraw in period 1, independent of the realization of their type.

Given a mechanism $\mathbf{m} = (c^1(1), \ldots, c^1(z), \ldots, c^1(N), c^2(0), \ldots, c^2(N-1))$ and a corresponding equilibrium, ex ante consumer welfare is defined as the sum of the expected utilities of the consumers. The mechanism that supports the symmetric constrained-efficient allocation requires that impatient consumers choose period 1 and patient consumers choose period 2. When impatient consumers choose period 1 and patient consumers choose period 2, we denote ex ante consumer welfare under mechanism $\mathbf{m}$ as $\hat{W}(\mathbf{m})$. Using the resource condition, (2), we can write ex ante consumer welfare as a function of $(c^1(1), \ldots, c^1(N-1))$, given by

$$\hat{W}(\mathbf{m}) = \sum_{\alpha=0}^{N-1} f(\alpha)\left[\sum_{z=1}^{\alpha} u(c^1(z)) + (N-\alpha)v\left(\frac{[Ny - \sum_{z=1}^{\alpha} c^1(z)]R}{N-\alpha}\right)\right]$$
$$+ f(N)\left[\sum_{z=1}^{N-1} u(c^1(z)) + u\left(Ny - \sum_{z=1}^{N-1} c^1(z)\right)\right]. \tag{3}$$

When all patient consumers choose period 1, we denote ex ante consumer welfare under mechanism $\mathbf{m}$ as $W^{\text{run}}(\mathbf{m})$, given by

$$W^{\text{run}}(\mathbf{m}) = \sum_{\alpha=0}^{N} f(\alpha)\left[\frac{\alpha}{N}\sum_{z=1}^{N} u(c^1(z)) + \frac{N-\alpha}{N}\sum_{z=1}^{N} v(c^1(z))\right]. \tag{4}$$

The so-called "optimal contract" will induce the patient consumers to choose period 2 and therefore must solve the following incentive compatibility constraint:

$$\sum_{\alpha=0}^{N-1} f_p(\alpha)\left[\frac{1}{\alpha+1}\sum_{z=1}^{\alpha+1} v(c^1(z))\right] \leq \sum_{\alpha=0}^{N-1} f_p(\alpha)v\left(\frac{[Ny - \sum_{z=1}^{\alpha} c^1(z)]R}{N-\alpha}\right). \tag{5}$$

Thus, the "optimal contract" solves[4]

$$\max_{(c^1(1),\ldots,c^1(N-1))} \hat{W}(\mathbf{m})$$
$$\text{subject to (5).} \tag{6}$$

We place quotation marks around "optimal contract" because we shall see that the solution to the planner's problem (6), which we denote as $\mathbf{m}^*$, could have a run equilibrium. However, (6) presupposes that the run equilibrium is never chosen. If the run equilibrium is chosen with positive probability, then $\mathbf{m}^*$ may not be optimal when the possibility of a run is taken into account. These issues are explored in Section IV, where we would call $\mathbf{m}^*$ the zero-optimal mechanism, referring to the situation in which the propensity to run is zero.

Let λ denote the Lagrangean multiplier on constraint (5). Then in

[4] Of course, one must check that the nonnegativity constraints are satisfied as well.

the planner's problem (6), the necessary conditions for an optimum are, for $\hat{\alpha} = 0, \ldots, N-1$,

$$\sum_{\alpha=\hat{\alpha}}^{N-1} f(\alpha)\left[u'(c^1(\hat{\alpha})) - Rv'\left(\frac{[Ny - \sum_{z=1}^{\alpha} c^1(z)]R}{N-\alpha}\right)\right]$$
$$+ f(N)[u'(c^1(\hat{\alpha})) - u'(c^1(N))]$$
$$+ \lambda\left\{\sum_{\alpha=\hat{\alpha}}^{N-1} f_p(\alpha)\left[v'\left(\frac{[Ny - \sum_{z=1}^{\alpha} c^1(z)]R}{N-\alpha}\right)\left(\frac{-R}{N-\alpha}\right)\right.\right.$$
$$\left.\left. - v'(c^1(\hat{\alpha}))\left(\frac{1}{\alpha+1}\right)\right]\right\} = 0 \qquad (7)$$

and

$$\lambda\left\{\sum_{\alpha=0}^{N-1} f_p(\alpha) v\left(\frac{[Ny - \sum_{z=1}^{\alpha} c^1(z)]R}{N-\alpha}\right)\right.$$
$$\left. - \sum_{\alpha=0}^{N-1} f_p(\alpha)\left[\frac{1}{\alpha+1}\sum_{z=1}^{\alpha+1} v(c^1(z))\right]\right\} = 0. \qquad (8)$$

Incentive compatibility when the other patient consumers choose period 2 is in general different from incentive compatibility when the other patient consumers choose period 1. If a patient consumer prefers to choose period 1 when other patient consumers choose period 1, we have a run equilibrium. Therefore, $\mathbf{m}^*$ might have a run equilibrium, which occurs when we have

$$\frac{1}{N}\sum_{z=1}^{N} v(c^1(z)) \geq v\left(\left[Ny - \sum_{z=1}^{N-1} c^1(z)\right]R\right). \qquad (9)$$

III. A Two-Consumer, I.I.D. Example

We now present an example for which we calculate the "optimal contract" $\mathbf{m}^*$ and show that the corresponding postdeposit game has a run equilibrium. There are two consumers, $N = 2$; each consumer is impatient with probability p and patient with probability $1 - p$. Types are uncorrelated. Let $c^1(1)$ be denoted by c. Then the expression for welfare simplifies to

$$\hat{W} = p^2[u(c) + u(2y - c)] + 2p(1-p)[u(c) + v((2y-c)R)]$$
$$+ 2(1-p)^2 v(yR). \qquad (10)$$

The incentive compatibility constraint (5) simplifies to

$$p\left[\frac{v(c)}{2}+\frac{v(2y-c)}{2}\right]+(1-p)v(c)\leq pv((2y-c)R)+(1-p)v(yR), \quad (11)$$

and the condition for a run equilibrium (9) simplifies to

$$\frac{v(c)}{2}+\frac{v(2y-c)}{2}\geq v((2y-c)R). \quad (12)$$

PROPOSITION 1. For some economies, a run equilibrium exists at the "optimal contract" $\mathbf{m}^*$.

Proof. Let the utility functions be given by

$$u(x)=\frac{Ax^{1-a}}{1-a}, \quad v(x)=\frac{x^{1-b}}{1-b}.$$

We shall find parameters A, a, b, p, R, and y for which the first-order conditions are necessary and sufficient for a solution to the planner's problem. Then we shall solve for the optimal mechanism and show that condition (12) holds. Let $A=10$, $a=1.01$, $b=1.01$, $p=\frac{1}{2}$, $R=1.05$, and $y=3$. The solution to the planner's problem[5] is given by

$$c=3.1481, \quad \lambda=4.0795. \quad (13)$$

Since the incentive compatibility constraint is binding and there is a single choice variable, c is found by solving (11), expressed as an equation rather than as an inequality.

It is easy to verify that, for these parameter values, the objective function is concave in c and the incentive compatibility constraint (left side minus right side) is increasing in c. Because strictly monotonic functions of a single variable are quasi-convex, it follows that the second-order conditions are satisfied. Thus (13) constitutes a solution to the planner's problem. The left side of (12) exceeds the right side, the difference being 0.000597, so $\mathbf{m}^*$ has a run equilibrium. Q.E.D.

The solution to the planner's problem (6), $\mathbf{m}^*$, is given by $c^1(1)=3.1481$, $c^1(2)=2.8519$; $c^2(0)=3.1500$, $c^2(1)=2.9945$.

There is a nonrun equilibrium of the postdeposit game in which all patient consumers choose period 2. The first impatient consumer would receive 3.1481 units of consumption in period 1, and the second impatient consumer would receive 2.8519 units in period 1. Thus we have partial suspension of convertibility, as in Wallace (1990) and Green and Lin (2000*a*, 2000*b*). Patient consumers receive 3.1500 units in period 2 if there are no impatient consumers and 2.9945 units in period 2 if

[5] The computations were performed using Maple V, release 5.1. Details are available from the authors.

there is one impatient consumer. Welfare, renormalized to be $\hat{W}+1{,}088$, is given by $\hat{W}(\mathbf{m}^*) = .27396$.

At the run equilibrium of the postdeposit game, consumers receive $c^1(1) = 3.1481$ with probability one-half, and they receive $c^1(2) = 2.8519$ with probability one-half. For these parameter values, renormalized welfare can be calculated as $W^{\text{run}}(\mathbf{m}^*) = .00519$.

Our example is very simple. There are only two consumers, and impatience is i.i.d. In Appendix A, we analyze an example with 300 consumers, with three possible realizations for α. Thus impatience is correlated across consumers, and each consumer is small relative to the market. For appropriately chosen parameters, we have as before that the optimal contract for $\hat{W}(\mathbf{m})$ also permits a run equilibrium.

We do not allow the bank to ask people to wait in line to declare themselves to be patient. We believe that to do so would be unrealistic. The question, then, is whether or not this is the source of run equilibria for $\mathbf{m}^*$. Appendix B answers this question in the negative. We redo our two-consumer, i.i.d. example to require all consumers to join in the queue in period 1 and declare themselves to be impatient or patient. For appropriately chosen parameters, the "optimal direct revelation mechanism" differs from the "optimal contract," but it also permits a run equilibrium.

IV. Sunspots and the Propensity to Run

Strictly speaking, run equilibria in Diamond and Dybvig (1983) are not equilibria at all, because consumers would not agree to the original contract if they knew that a run would take place (see Postlewaite and Vives 1987). Diamond and Dybvig suggest that a run could take place in equilibrium with positive probability, triggered by some extrinsic random variable "sunspots," as long as the probability of the run is sufficiently small. Here we formalize this notion by defining the predeposit game and calculate what "sufficiently small" is for an example.[6]

Here is the timing of the predeposit game, which takes place after the bank announces its mechanism. In period 0, consumers decide whether or not to deposit.[7] At the beginning of period 1, each consumer learns her type after observing a sunspot variable, σ, distributed uni-

[6] Cooper and Ross (1998), restricting themselves to simpler contracts, also model runs being triggered by sunspots.

[7] A consumer could invest her endowment herself instead of dealing with the bank. However, we do require that unharvested "trees" cannot be traded. This is to rule out the case in which a patient depositor (claiming to be impatient) trades period 1 consumption withdrawn from the bank for unharvested trees. Jacklin (1987) has shown that such a market undermines the optimal contract, and his argument applies to our setting as well. Ruling out this asset market is merely to posit that only banks can provide the liquidity necessary to pay for period 1 consumption (see Diamond 1997).

formly on [0, 1].[8] Sunspots do not affect preferences, the likelihood of being impatient, endowments, or technology. Now the period in which a consumer arrives can depend on the realization of the sunspot variable σ as well as the realization of her type. We assume that the bank cannot choose a withdrawal schedule that directly depends on σ.[9] To facilitate the comparison between the predeposit game and the postdeposit game, we take the space of mechanisms, M, to be the same in both games.[10]

DEFINITION 2. Given a mechanism $\mathbf{m} \in M$, the predeposit game is said to have a *run equilibrium* if there is a subgame-perfect Nash equilibrium in which (i) consumers are willing to deposit, and (ii) for some set of realizations of σ occurring with positive probability, all consumers choose to withdraw in period 1, independent of the realization of their type.

PROPOSITION 2. Consider a mechanism $\mathbf{m} \in M$, for which the postdeposit game has an equilibrium in which all patient consumers choose period 2, yielding welfare strictly higher than welfare under autarky.[11] Then the predeposit game has a run equilibrium if and only if the postdeposit game has a run equilibrium.

Proof. Let the predeposit game have a run equilibrium under the mechanism $\mathbf{m}$. Then for some realizations of σ, all consumers choose period 1 in the subgame after deposits are made and σ is observed. Since this subgame must be in equilibrium and since the subgame is identical to the postdeposit game, there must be an equilibrium to the postdeposit game in which all consumers choose period 1. Thus the postdeposit game has a run equilibrium.

Let the postdeposit game have a run equilibrium under the mechanism $\mathbf{m}$. Construct a run equilibrium to the predeposit game as follows. First, consumers deposit their endowment. Next, consumers withdraw as follows. For $\sigma < s$, all consumers choose period 1. For $\sigma \geq s$, impatient consumers choose period 1 and patient consumers choose period 2. Each subgame, after deposits are made and σ is observed, is in equilibrium. It is an equilibrium for all consumers to choose period 1 when we have $\sigma < s$ because the postdeposit game has a run equilibrium. It is an equilibrium for impatient consumers to choose period 1 and pa-

[8] Uniformity is assumed without loss of generality.

[9] The bank cannot observe the event triggering the run. Either the bank cannot observe the sunspot variable itself or it does not know which values of σ will trigger a run equilibrium (if one exists).

[10] Strictly speaking, a mechanism for the predeposit game should specify outcomes as a function of the number of depositors. For the predeposit game we interpret $\mathbf{m} \in M$ as providing autarky consumption (y in period 1 and Ry in period 2) unless all N consumers deposit. Introducing more complicated mechanisms of the form $m(n)$ does not change our results and does not appear to add further insights, so we avoid this complication.

[11] Under autarky, an impatient consumer receives y units of consumption, and a patient consumer receives Ry units of consumption.

tient consumers to choose period 2 when we have $\sigma \geq s$ because the postdeposit game is assumed to have an equilibrium in which all patient consumers choose period 2. Finally, for sufficiently small s, consumers are willing to deposit. The reason is that overall ex ante welfare is $sW^{\text{run}}(\mathbf{m}) + (1 - s)\hat{W}(\mathbf{m})$, where W^{run} is welfare in the run equilibrium and $\hat{W}$ is welfare in the no-run equilibrium. For sufficiently small s, welfare strictly exceeds welfare under autarky, so each consumer is willing to deposit if other consumers deposit. Q.E.D.

If the planner is unable to choose the equilibrium he likes, the truly optimal mechanism should depend on how consumers select among multiple equilibria to the postdeposit games. Since we are looking for optimal mechanisms, we restrict attention to mechanisms with an equilibrium in which all patient consumers choose period 2. We suppose that the economy has a *propensity to run* in the following sense. Whenever we have $\sigma < s$, all consumers choose to arrive at the bank in period 1, whenever the postdeposit game admits a run equilibrium. If the postdeposit game does not have a run equilibrium, then all patient consumers choose period 2. When we have $\sigma \geq s$, the equilibrium is selected in which all patient consumers wait for the second period. Such an economy is said to have a propensity to run s. The fully optimal mechanism now depends on the parameter s.

Definition 3. Given a mechanism $\mathbf{m}$ and a propensity to run s, ex ante welfare for the predeposit game, denoted as $W(\mathbf{m}, s)$, is given by

$$W(\mathbf{m}, s) = \tag{14}$$

$$\begin{cases} sW^{\text{run}}(\mathbf{m}) + (1 - s)\hat{W}(\mathbf{m}) & \text{if } \mathbf{m} \text{ has a run equilibrium} \\ \hat{W}(\mathbf{m}) & \text{if } \mathbf{m} \text{ does not have a run equilibrium.} \end{cases}$$

The mechanism that maximizes $W(\mathbf{m}, s)$ subject to the incentive compatibility constraint (5) is called the *s-optimal mechanism.*

We now show that, for the example of Section III, the s-optimal mechanism has a run equilibrium for sufficiently small s. Furthermore, we shall calculate the cutoff value of s below which the s-optimal mechanism has a run equilibrium. This formalizes the idea that, if the probability of a run is sufficiently small, the optimal mechanism tolerates bank runs. Altering the mechanism to eliminate the possibility of a run leads to lower welfare.

Proposition 3. For some economies with a sufficiently small propensity to run, s, the optimal mechanism for the predeposit game has a run equilibrium.

Proof. Consider the predeposit game for the economy of Section III. Since patient consumers choose period 2 when $\sigma \geq s$, incentive compatibility condition (11) must hold. It is shown in Section III that (11) holds as an equality at the optimal mechanism to the postdeposit game,

TABLE 1

The "Optimal Contract" $\mathbf{m}^*$	
$c^1(1) = 3.1481$	$c^1(2) = 2.8519$
$c^2(0) = 3.1500$	$c^2(1) = 2.9945$
Best Mechanism Immune from Runs: $\mathbf{m}^{\text{no-run}}$	
$c^1(1) = 3.1463$	$c^1(2) = 2.8537$
$c^2(0) = 3.1500$	$c^2(1) = 2.9964$

which is the zero-optimal mechanism for the predeposit game. Thus incentive compatibility must bind and (11) must hold as an equality for sufficiently small s, by continuity. It follows that for sufficiently small s, the s-optimal mechanism is characterized by the unique c solving (11), which is $\mathbf{m}^*$, calculated in Section III. By continuity, $W(\mathbf{m}, s)$ can be made arbitrarily close to .27396 for sufficiently small s, which exceeds welfare under autarky, .066841. Thus consumers are willing to deposit. Since $\mathbf{m}^*$ has a run equilibrium for the postdeposit game, it also has a run equilibrium for the predeposit game. Q.E.D.

For general economies, computation of the s-optimal mechanism might be difficult. Even if incentive compatibility binds and s is small, the s-optimal mechanism might depend on s (and differ slightly from $\mathbf{m}^*$). The proof of proposition 3 is simplified considerably by the fact that a mechanism for the example of Section III is characterized by the single variable c. A binding incentive compatibility constraint completely pins down the mechanism, so the s-optimal mechanism is independent of s for small s. Given the structure of our example, we can determine the s-optimal mechanism for all $s \in [0, 1]$ as follows.

For small s, the s-optimal mechanism for our example economy is $\mathbf{m}^*$, as argued in the proof of proposition 3. As s increases, the welfare under $\mathbf{m}^*$ falls, because a bank run is more and more likely to occur in equilibrium. Eventually, the propensity to run becomes high enough so that $\mathbf{m}^*$ is inferior to the best mechanism that is immune to runs, characterized by the unique c for which (12) holds as an equality.[12] We denote this mechanism as $\mathbf{m}^{\text{no-run}}$. The two mechanisms are compared in table 1.

Under $\mathbf{m}^*$, when the system is working appropriately and there is no bank run, welfare $\hat{W}(\mathbf{m}^*)$ is .27396. However, the system is fragile, and when a run occurs, welfare $W^{\text{run}}(\mathbf{m}^*)$ is .00519. Under $\mathbf{m}^{\text{no-run}}$, the system is immune from runs, and the unique equilibrium is for the patient consumers to choose period 2; welfare $\hat{W}(\mathbf{m}^{\text{no-run}})$ is .27158. When the

[12] Since welfare is increasing in c for all c satisfying inequality (12), the best mechanism immune to runs must satisfy (12) as an equality.

propensity to run is small enough, the s-optimal mechanism overall will be $\mathbf{m}^*$, since with high probability consumers select the equilibrium in which patient consumers wait until period 2. The small probability of a run does not warrant the discrete reduction in $c^1(1)$, so that the condition for a run equilibrium, (12), is not satisfied. However, if the propensity to run crosses a threshold, the optimal mechanism switches to $\mathbf{m}^{\text{no-run}}$.[13] We are now in a position to calculate the cutoff value, s_0, where the s-optimal mechanism is $\mathbf{m}^*$ for $s < s_0$, and the s-optimal mechanism is $\mathbf{m}^{\text{no-run}}$ for $s > s_0$.

For our example, we can calculate the largest propensity to run consistent with the s-optimal mechanism having a run equilibrium to the predeposit game. Solving

$$(1 - s_0)\hat{W}(\mathbf{m}^*) + s_0 W^{\text{run}}(\mathbf{m}^*) = \hat{W}(\mathbf{m}^{\text{no-run}})$$

yields $s_0 = .008848$, so the economy will tolerate the occurrence of a bank run with probability .008848. That is, if consumers have a propensity to run below .008848, the optimal mechanism accepts this, and the equilibrium probability of a run, *at the optimal mechanism,* is .008848. On the other hand, if consumers have a propensity to run above .008848, the s-optimal mechanism is immune from runs because the stronger incentive compatibility constraint is imposed (i.e., [12] holds as an equality). This trade-off between fragility and efficiency is depicted in figure 1. The downward-sloping line depicts ex ante welfare based on randomizing over the run and the no-run equilibria to $\mathbf{m}^*$. The horizontal line depicts welfare based on the best contract immune from runs, $\mathbf{m}^{\text{no-run}}$. Welfare at the s-optimal mechanism, as a function of s, is given by the upper envelope (shown in bold) of the two lines in figure 1.

All these calculations are based on a special assumption about consumer beliefs, as reflected in our notion of propensity to run. Consumers expect a run with (exogenous) probability s if and only if the mechanism has a run equilibrium. Other rational expectations are possible. For example, the probability of a run might depend on the magnitude of the incentive to choose period 1. If a patient consumer has only a slight preference for period 1 during a run, then the probability of a run might be reduced. In general, the propensity to run could depend on the mechanism, which could enrich the problem of finding the optimal contract.

[13] A third possibility must be considered in which the mechanism admits a run equilibrium but incentive compatibility does not bind and inequality (11) is strict. This is conceivable because relaxing (11) is beneficial when a run occurs. However, for our example, any such mechanism is dominated by $\mathbf{m}^*$ unless s is close to one, in which case $\mathbf{m}^{\text{no-run}}$ is superior.

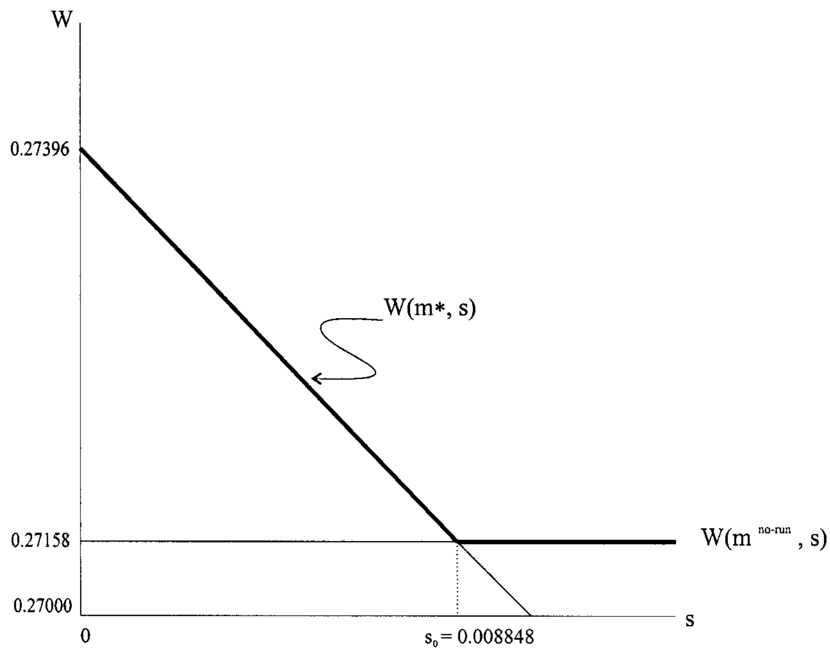

FIG. 1.—The s-optimal mechanism: welfare (in bold) as a function of the propensity to run.

This discussion is related to the literature on financial fragility. The costs associated with occasional equilibrium bank runs are the downside of financial fragility in our model. However, when the system is working smoothly and the equilibrium in which patient consumers choose period 2 is selected, a fragile system is more efficient. When the propensity to run is below s_0, this trade-off leads to tolerating a fragile system. When the propensity to run is above s_0, the cost of financial fragility is too high, and it is better to establish a stable system. See the papers by Allen and Gale (2000) and Lagunoff and Schreft (2001) for an analysis of financial crises based on local interactions. See also Champ, Smith, and Williamson (1996) for an analysis of banking panics and Kiyotaki and Moore (1997) for a study of credit cycles.

V. Concluding Remarks

We have shown that the possibility of equilibrium bank runs does not depend on a simple and suboptimal specification of the deposit contract or mechanism. There are economies in which the optimal mechanism, within a broad class that includes suspension schemes, induces a postdeposit game with a run equilibrium. To eliminate this run equilibrium would require a sacrifice of welfare, as compared to the equilibrium in which the patient consumers wait. By introducing sunspots that trigger the bank run, we construct an example in which the optimal mechanism tolerates a positive probability of a run. We calculate, for our simple example, the highest probability of a run that can be tolerated, above which the planner should take steps to eliminate runs. Equilibrium bank runs are consistent with large economies and correlated types.

Which types of economies will tolerate runs? One requirement is that there be significant uncertainty about the aggregate number of impatient and patient consumers. If α were known, suspension of convertibility would eliminate the run equilibrium while preserving the no-run equilibrium.[14] A second requirement is that the utility functions reflect a high degree of "impulse demand" by the impatient consumers, relative to R. The impatient must be well treated at the constrained-efficient allocation, providing the temptation to join a run. A third requirement is that a patient consumer's incentive to choose period 1 when other patient consumers choose period 1 is greater than a patient consumer's incentive to choose period 1 when other patient consumers choose period 2. Unfortunately, translating this condition on the optimal contract to a condition on the parameters of the economy is difficult. Our simulations indicate that it is most likely to be satisfied when the coefficients of relative risk aversion are small in absolute value.

One factor that might be thought to work in favor of tolerating runs is the specification that impatient consumers "die" after period 1. If impatient consumers had a "consumption opportunity" in period 1 but received utility from consumption in period 2 as well, the bank would have additional leverage because most of the resources would be held until period 2. In an earlier version of this paper (Peck and Shell 1999), we analyzed a model in which there are two technologies and the bank can hold only the more liquid asset. We imposed additional restrictions on the mechanism but continued to allow suspension schemes and showed that equilibrium bank runs always exist in that framework.

The sophisticated contracts studied here and in Green and Lin (2000*a*, 2000*b*) are apparently not observed in practice. One explanation is that we ignore possible moral hazard problems faced by the bank. See Calomiris and Kahn (1991) for an explicit analysis of moral hazard and embezzlement in banking. Also, in our model, the choice of when to arrive replaces the requirement to report one's type. If we introduced an indivisibility in period 1 consumption opportunities, as in Peck and

[14] For utility functions satisfying $v(0) = -\infty$, we require $f(N-1) > 0$ or $f(N) > 0$, or else the $N-$ 1st consumer could be given zero period 1 consumption, which would not matter in the no-run equilibrium. No patient consumer would join a run and face a positive probability of infinite punishment.

Shell (1999), then the equilibrium contracts would indeed be fairly simple. Further research is needed on this issue, but the present paper indicates that the possibility of run equilibrium does not melt away when more complicated contracts can be introduced.

Appendix A

Robustness to Many Consumers and Correlated Types

To show that a run equilibrium at the optimal mechanism extends beyond two consumers and i.i.d. types, here we construct another example with many consumers and correlated types. In particular, we consider an example with three possible realizations of α.

EXAMPLE 2.

$$N = 300, \quad y = 5, \quad R = 1.05, \quad f(100) = \tfrac{1}{3},$$

$$f(200) = \tfrac{1}{3}, \quad f(300) = \tfrac{1}{3}, \quad f_p(100) = \tfrac{2}{3}, \quad f_p(200) = \tfrac{1}{3},$$

$$f_p(300) = 0, \quad u(x) = -100x^{-1}, \quad v(x) = -x^{-1}.$$

Not surprisingly, the "optimal contract" provides the same period 1 consumption for each of the first 100 consumers, each of the second 100 consumers, and each of the third 100 consumers. Thus let $c^1(z) \equiv c^{1,1}$ for $z = 1, \dots, 100$, let $c^1(z) \equiv c^{1,2}$ for $z = 101, \dots, 200$, and let $c^1(z) \equiv c^{1,3}$ for $z = 201, \dots, 300$. The necessary first-order conditions, (7) and (8), can be solved for $c^{1,1}$ and $c^{1,2}$, yielding

$$c^{1,1} = 5.05955, \quad c^{1,2} = 5.29658, \quad \lambda = 3{,}899.3. \tag{A1}$$

From (A1), we see that the incentive compatibility constraint is binding. The entire mechanism is determined from (A1). For the equilibrium in which the patient consumers wait, consumptions are[15]

$$c^{1,1} = 5.05955, \quad c^{1,2} = 5.29658, \quad c^{1,3} = 4.64387,$$
$$c^2(100) = 5.21873, \quad c^2(200) = 4.87606.$$

At the optimal contract, patient consumers are indifferent between choosing period 1 and choosing period 2 when all other patient consumers choose period 2. However, when all other patient consumers choose period 1, it turns out that the remaining patient consumer strictly prefers to choose period 1. In other words, this mechanism supports a run equilibrium to the postdeposit game.

[15] It turns out that the incentive compatibility constraint is not quasi-convex for this problem. However, we can verify that (A1) determines the optimal mechanism. First, we analytically solve the incentive compatibility constraint for $c^{1,2}$ as a function of $c^{1,1}$. Next, substitute into the expression for W to get welfare as a function of $c^{1,1}$ only. This function is concave and is maximized at $c^{1,1} = 5.05955$. Finally, contradict the supposition that there can be a solution in which incentive compatibility does not bind.

Appendix B

Direct Revelation Mechanisms

Here we adapt the postdeposit game to allow the planner to choose direct revelation mechanisms. As in Green and Lin (2000*a*, 2000*b*), consumers arrive at the bank in period 1 and report whether they are impatient or patient. The sequential service constraint requires period 1 consumption to be independent of the reports of those arriving later in the queue. However, this sequential service constraint is different from that assumed earlier. For example, now someone first in the queue and reporting "impatient" can receive a different level of period 1 consumption from someone second in the queue and reporting "impatient" after the first person reports "patient." The model defined in Section II cannot make this distinction because someone wishing to report "patient" must do this in period 2. The queue in period 1 consisted only of those wishing to receive consumption in period 1.

The following example is the same as that of Section III, with slightly different parameters. There are two consumers, $N = 2$; each consumer is impatient with probability p and patient with probability $1 - p$. Types are uncorrelated. A mechanism specifies period 1 consumption as a function of the history of reported types and period 2 consumption as a function of a consumer's position in the period 1 queue and the sequence of reported types. This specification builds in the appropriate sequential service constraint. The mechanism that maximizes welfare subject to resource and incentive compatibility constraints must satisfy the following conditions. Consumers who report "impatient" receive no consumption in period 2. Consumers who report "patient" receive no consumption in period 1, and if both consumers report "patient," they each receive the same consumption, yR, in period 2. Thus we can characterize the optimal mechanism by the period 1 consumption when the consumer first in the queue reports "impatient," denoted by c, and the period 1 consumption when the second consumer in the queue reports "impatient" after the first consumer reports "patient," denoted by $\hat{c}$. Thus nonzero consumptions are given by

$$\begin{aligned} c^1(I) &= c, \\ c^1(P, I) &= \hat{c}, \\ c^1(I, I) &= 2y - c, \\ c^2(I, P) &= (2y - c)R, \\ c^2(P, I) &= (2y - \hat{c})R, \\ c^2(P, P) &= yR. \end{aligned}$$

The expression for welfare simplifies to

$$\begin{aligned} W = {} & p^2[u(c) + u(2y - c)] + p(1 - p)[u(c) + v((2y - c)R)] \\ & + p(1 - p)[u(\hat{c}) + v((2y - \hat{c})R)] + 2(1 - p)^2 v(yR). \end{aligned} \tag{B1}$$

The incentive compatibility constraint simplifies to

$$\frac{v(c)}{2}+\frac{pv(2y-c)+(1-p)v(\hat{c})}{2}\leq\frac{1}{2}[pv((2y-\hat{c})R)+(1-p)v(yR)]$$
$$+\frac{1}{2}[pv((2y-c)R)+(1-p)v(yR)], \qquad \text{(B2)}$$

and the condition for a run equilibrium simplifies to

$$\frac{v(c)}{2}+\frac{v(2y-c)}{2}\geq\frac{v((2y-\hat{c})R)}{2}+\frac{v((2y-c)R)}{2}. \qquad \text{(B3)}$$

Let the utility functions be given by

$$u(c) = \frac{Ac^{1-a}}{1-a}, \quad v(c) = \frac{c^{1-b}}{1-b},$$

and let $A = 10$, $a = 2$, $b = 2$, $p = \frac{1}{2}$, $R = 1.05$, and $y = 3$. The planner's problem is to choose c and $\hat{c}$ to maximize (B1) subject to the incentive compatibility constraint (B2). The solution is given by

$$\hat{c} = 3.20115, \quad c = 3.09395, \quad \lambda = 1.94897. \qquad \text{(B4)}$$

Although (B2) is not necessarily quasi-convex, we can show that (B4) constitutes a solution. We know that (B2) must hold as an equality because welfare in (B1) is increasing in c and $\hat{c}$ within the relevant ranges and because the incentive compatibility constraint is downward sloping in $c-\hat{c}$ space. Given that (B2) must hold as an equality, we can analytically solve (B2) for $\hat{c}$ as a function of c. Substituting for $\hat{c}$ in the welfare expression (B1), we transform the problem into the unconstrained maximization of welfare, as a function of c. This problem is concave and is maximized at $c = 3.09395$.

From (B4), we determine the optimal direct revelation mechanism as

$$\begin{aligned} c^1(I) &= 3.09395,\\ c^1(P, I) &= 3.20115,\\ c^1(I, I) &= 2.90605,\\ c^2(I, P) &= 3.05135,\\ c^2(P, I) &= 2.93879,\\ c^2(P, P) &= 3.15000. \end{aligned}$$

In the equilibrium in which consumers report truthfully, welfare is -3.58303. Inequality (B3) holds as well, which implies that there is a run equilibrium in which all consumers claim to be impatient. In the run equilibrium, one consumer receives period 1 consumption of 3.09395, and the other consumer receives period 1 consumption of 2.90605.

The parameters in this example are identical to the parameters in the example of Section III, except that here we have $a = b = 2$, whereas in Section III we have $a = b = 1.01$. Having a risk aversion parameter of two is empirically plausible and allows for an analytic solution. However, when we consider $a = b = 2$ in the example of Section III, mechanism $\mathbf{m}^*$ does not have a run equilibrium. For the example of this Appendix, the optimal mechanism has a run equilibrium when we allow direct revelation mechanisms, but not when a consumer's strategy is to choose a round. Thus the set of economies for which the "optimal" mechanism admits a run equilibrium does not shrink when we allow for direct revelation mechanisms.[16]

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