How Optimal Banking Contracts Tolerate Runs*

Karl Shell
Cornell University

Yu Zhang
Xiamen University

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Abstract

We perform comparative statics for a 2-depositor banking model defined by the scalar c – the withdrawal of the first in line in period 1. In the *post-deposit* game, the *unconstrained efficient allocation* is strongly implementable, or weakly implementable, or not implementable. Bank runs matter in the last two cases. In the *pre-deposit* game the optimal contract $c^*(s)$ is weakly decreasing in the run probability s until it levels off at the best run-proof c. If the incentive compatibility constraint (ICC) does not bind, then $c^*(s)$ is *strictly* decreasing in s until it levels off at the best run-proof contract. Then the optimal allocation in the *pre-deposit* game is never a mere randomization over the *unconstrained efficient allocation* and the corresponding run allocation from the *post-deposit* game. If the ICC binds, however, then $c^*(s)$ is constant for small s and the optimal allocation in the *pre-deposit* game is then for small s a randomization over the *constrained efficient allocation* and the corresponding run allocation from the *post-deposit* game.

Keywords: bank runs, constrained efficient allocation, deposit contract, impulse demand, pre-deposit game, post-deposit game, run probability, strategic complementarity, sunspots, unconstrained efficient allocation

JEL classification numbers: G21; E44

1 Introduction

Bryant (1980) and Diamond and Dybvig (1983) – hereafter DD – introduced the modern literature on panic-based bank runs. The bank deposit contract is a mechanism designed to improve the welfare of depositors facing an uncertain impulse demand (i.e., when they become impatient). Since the impulse demand itself is not directly observable, it is uninsurable in the market. The deposit contract facilitates some "insurance" by specifying early and late withdrawals such that they are (Bayesian) incentive compatible: depositors with different liquidity needs correctly self-select their types, i.e., an impatient depositor chooses early withdrawal and a patient depositor chooses late withdrawal if he expects that the other patient depositors will also choose late withdrawal. Besides the "good" Bayes-Nash equilibrium in which the depositors self-select, DD show that there is also a "bad" equilibrium, the self-fulfilling bank run. When a bank run occurs, depositors attempt to withdraw early independent of their liquidity needs. Bank runs are possible because, the "good" equilibrium is only weakly implementable rather than strongly implementable.[1] If the patient depositor expects that a bank run will take place, he will choose early withdrawal.

But given the two equilibria of the *post-deposit* game, the consumers will not deposit if they anticipate the run: the bank run will not be an equilibrium for the *pre-deposit* game. DD seem to have recognized this problem and offer sunspots as an answer.[2] Peck and Shell (2003) – hereafter PS – examine *sunspot equilibrium* in the *pre-deposit* game.[3] There is also intrinsic

[1] In other words, for a patient depositor, choosing late withdrawal is Bayesian incentive compatible rather than dominant-strategy incentive compatible.

[2] In their paper (page 410), DD mention that "This could happen if the selection between the bank run equilibrium and the good equilibrium depended on some commonly observed random variable in the economy. This could be a bad earnings report, a commonly observed run at some other bank, a negative government forecast, or *even sunspots* (emphasis ours)."

Postlewaite and Vives (1987) show how bank runs can be seen as a prisoner's dilemma-type situation in which there is a unique equilibrium that involves a positive probability of a run.

[3] See also Cooper and Ross (1998) and Ennis and Keister (2006). These two papers analyze how banks respond to the possibility of runs in their design of deposit contracts and in their investment decisions. Gu (2011) analyzes noisy sunspots and bank runs.

uncertainty (as opposed to extrinsic uncertainty in the form of sunspots) in the PS model: (1) The aggregate number of impatient consumers is uncertain. It could be $0, 1$ or 2. This is important because it does not allow the bank to know with certainty that a run is underway. (2) Sequential service is taken seriously (Wallace (1988)). When 2 depositors withdraw early, their positions in the queue are random.

PS show that a *sunspot-driven* run can be an equilibrium in the *pre-deposit* game as long as (1) the *post-deposit* game has both a run equilibrium and a non-run equilibrium, and (2) the run probability is below a threshold level. PS use a 2-depositor example[4] to formalize the threshold probability, and the optimal deposit contract. In the example, the banking contract is characterized by c which is the withdrawal of the first in line in period 1. The optimal c is denoted by $c^*(s)$ which is a function of the exogenous, sunspot probability s.[5] In the PS example, $c^*(s)$ is a step function: If the probability s is less than the threshold probability s_0, the optimal contract $c^*(s)$ tolerates runs and is a constant. If the probability s is greater than s_0, the optimal contract is the best run-proof contract.

In this paper, we ask: Why doesn't the optimal contract become more conservative as the run probability increases (until runs are no longer tolerated)? In other words, shouldn't $c^*(s)$ be strictly decreasing in s before it switches to the best run-proof contract? If yes, in which economies will we have this property and in which economies is $c^*(s)$ a step function? These issues are important to banks and regulators[6]. Contracts and regulations could well be different based on the economy's level of fragility or pessimism as indicated by the probability s. To answer this question and keep the analysis

These three papers focus on simple deposit contracts, while PS allows for partial or full suspension of convertibility.

[4]The example is in their proof for Proposition 3 (Page 114).

[5]Like other papers in the literature, PS assume that the bank chooses the contract that maximizes the ex ante expected utility of depositors. This assumption can be justified when banks compete for deposits, there is no entry cost for banks, and the depositors are ex ante identical.

[6]It is also an important question in the theory of sunspot equilibrium. Some practitioners confuse sunspot equilibrium (SSE) with randomizations over certainty equilibria (CE). Not all SSE are randomizations over CE. Not all randomizations over CE are SSE. See Shell (2008).

tractable, we employ the 2-depositor banking example in PS.[7] Instead of relying solely on numerical examples, we provide the global comparative statics for this economy.

We characterize the parameters for which there are both a non-run equilibrium and also a run equilibrium to the optimal contract in the *post-deposit* game. When there are multiple equilibria in the *post-deposit* game, the optimal contract in the *pre-deposit* game will reflect two concerns: (1) maintaining the non-run equilibrium (i.e., satisfying the ICC) and (2) balancing the trade-off between the non-run and run equilibria. If the *unconstrained efficient allocation*[8] is strongly implementable, neither of these two concerns arise. If the *unconstrained efficient allocation* is weakly implementable, maintaining the non-run equilibrium does not play a role and the optimal contract will be solely determined by balancing the trade-off between the non-run and run equilibria. Hence the optimal contract $c^*(s)$ is continuous and strictly decreasing until it switches to the best run-proof c. The optimal contract is never a mere randomization over the two allocations from the post-deposit game.

If the *unconstrained efficient allocation* is not implementable, at least when the run probability s is small, maintaining the non-run equilibrium determines the optimal contract. When it occurs, the optimal contract does not respond to changes in s since the binding ICC is independent of s. This is why a larger run probability does not induce a more conservative contract even when s is small. For small s, the allocation is a mere randomization over the equilibrium allocations for the post-deposit game.

[7] Green and Lin (2000), Andolfatto, Nosal and Wallace (2007) and Nosal and Wallace (2009) analyze a model similar to PS. The main differences among the models are on the amount of information that a depositor has at the time of making his withdrawal decision. Ennis and Keister (2009) show that the PS assumptions on marginal utilities are not necessary for the qualitative results in PS. Ennis and Keister (2009) also study the Green-Lin model under a more general specification of the distribution of types across agents. See Ennis and Keister (2010) for a good survey on this part of the literature.

[8] The *unconstrained efficient allocation* is the best allocation that can be attained when agent types (patient or impatient) are observable. In other words, the allocation maximizes the ex ante expected utility of agents without imposing incentive compatibility, but it is still subject to the sequential service and resource feasibility constraints. See Ennis and Keister (2010). The associated contract is sometimes called the "first-best contract".

In the next section, we introduce the notation and the setup. In Section 3, we analyze the *post-deposit* game and characterize the parameters such that the non-run equilibrium and the run equilibrium coexist in the *post-deposit* game. In Section 4, we provide the global comparative statics for the optimal contract. We focus on how different values of the parameter describing the relative strength of the impulse demand lead to one of three cases: Depending on the parameters, the *unconstrained efficient allocation* is (1) strongly implementable, (2) weakly implementable, or (3) not implementable. We describe the optimal contract for each of the three cases. In Appendix 1, we provide the proofs of our results. In Appendix 2, we provide comparative statics with respect to the other parameters.

2 The Environment

The notation and the setup of the model is the same as in PS. There are two consumers and three periods: 0, 1 and 2. In period 0, each consumer is endowed with y units of the consumption good.[9] Let c^1 and c^2 denote the withdrawals of the depositor in period 1 and 2 respectively. The impatient consumers derive utility only from period-1 consumption while the patient consumers derive utility only from period-2 consumption. The patient consumers can store consumption goods costlessly across the two periods. The impatient and patient consumers, respectively, receive utilities $u(c^1)$ and $v(c^1 + c^2)$, where

$$u(x) = A\frac{(x)^{1-a}}{1-a}, \text{ where } A > 0. \tag{1}$$

$$v(x) = \frac{(x)^{1-b}}{1-b} \tag{2}$$

A reflects the strength of the "impulse demand" of impatient consumers. We will discuss this parameter in detail in section 3. The parameters a and b, both larger than 1, are the coefficients of relative risk aversion of the impatient and patient consumers respectively. Consumers are identical in period

[9] There are no endowments in periods 1 and 2.

0. In period 1, each consumer becomes either impatient with probability p or patient with probability $1-p$. Types are uncorrelated and private information. Since the number of consumers is finite, the aggregate number of patient depositors is stochastic. In period 1, each depositor also observes a sunspot variable δ distributed uniformly on $[0,1]$. Besides the storage technology, there is another investment technology. Investing one unit of period 0 consumption yields $R>1$ units if harvested in period 2 and yields 1 unit if harvested in period 1.

The sequential service constraint is part of the physical environment. A depositor visits the bank only when he makes a withdrawal. When a depositor learns his type and makes his withdrawal decision, he does not know his position in the bank queue. If more than one depositor chooses to withdraw, a depositor's position in the queue is random; positions in the queue are equally probable.

3 Post-Deposit Game

In the *post-deposit* game, $c \in [0,2y]$ completely defines the corresponding allocation. From PS, we know that the *pre-deposit* game has a run equilibrium only if the *post-deposit* game has both a non-run equilibrium and a run equilibrium.[10] In other words, given an arbitrary feasible contract $c \in [0,2y]$, the *pre-deposit* game has a run equilibrium only if

$$[v(c)+v(2y-c)]/2 > v[(2y-c)R] \tag{3}$$

and

$$pv[(2y-c)R]+(1-p)v(yR) \geq p[v(c)+v(2y-c)]/2+(1-p)v(c). \tag{4}$$

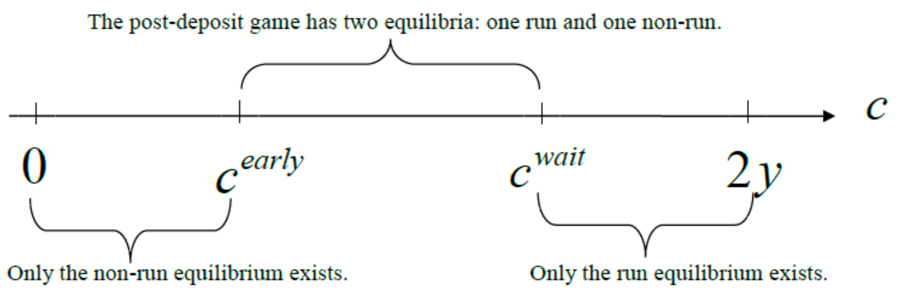
Inequality (3) tells us that a patient depositor strictly prefers early withdrawal when he expects that the other depositor, if patient, will also choose

[10] Proposition 2 in PS.

early withdrawal. Inequality (4), which is the ICC, tells us that a patient depositor (weakly) prefers late withdrawal when he expects that the other depositor, if patient, will also choose late withdrawal.[11]

Let c^{early} and c^{wait} be the values of c such that inequalities (3) and (4) hold as equalities. If $c^{early} < c^{wait}$, these two values become thresholds which determine a patient depositor's withdrawal strategies in the *post-deposit* game: c^{early} is the threshold of c *beyond* which a patient depositor chooses early withdrawal if he expects that the other patient depositor will also choose early withdrawal. c^{wait} is the threshold of c *below* which a patient depositor chooses late withdrawal if he expects that the other patient depositor will also choose late withdrawal. Therefore, in the *post-deposit* game, we have a unique non-run equilibrium for $c \in [0, c^{early}]$, two equilibria (one non-run equilibrium and one run equilibrium) for $c \in (c^{early}, c^{wait}]$, and a unique run equilibrium for $c \in (c^{wait}, 2y]$. See Figure 1.

Figure 1. Equilibrium in the Post-Deposit Game



If $c^{early} \geq c^{wait}$, we have a unique non-run equilibrium for $c \in [0, c^{wait}]$ and a unique run-equilibrium for $c \in (c^{early}, 2y]$. For $c \in (c^{wait}, c^{early}]$, a patient depositor chooses early (late) withdrawal if he expects that the other patient depositor will choose late (early) withdrawal and, therefore, a unique partial-bank-run equilibrium exists. Hence, if $c^{early} \geq c^{wait}$, for any $c \in [0, 2y]$ the equilibrium in the *post-deposit* game is unique. In the next section, we will see that the optimal contract is straightforward if $c^{early} \geq c^{wait}$ and full bank run cannot be tolerated by the optimal contract.

[11] As in other papers in the literature, we assume that a patient depositor chooses early withdrawal if he *strictly* prefers the early withdrawal to the late withdrawal. And he chooses late withdrawal if he *weakly* prefers to do so.

In most of the paper, we focus on the leading case $c^{early} < c^{wait}$, which requires the parameters b and R to satisfy the condition specified in Lemma 1. The intuition of Lemma 1 is the following. From inequality (3), the patient depositor's comparison between the early and late withdrawals depends on: (1) his attitude toward the risk of being the second in line when participating in runs, and (2) the productivity of the investment technology R. Given R, the more risk-averse the patient depositor, the less preferable is it for the patient depositor to run on the bank. Given his attitude toward risk, the more productive the investment, the less preferable is it for the patient depositor to run on the bank. Since $v(c) = (c^{1-b} - 1)/(1 - b)$, a patient depositor's attitude toward risk is characterized by b. Hence the existence of $c \in [0, 2y]$ satisfying inequality (3) restricts the parameters b and R.[12] The same is true for inequality (4).

Lemma 1 $c^{early} < c^{wait}$ *if and only if*

$$b < \min\{2, 1 + \ln 2/ \ln R\} \tag{5}$$

Given R and b satisfying (5), we know that a run-equilibrium exists in the *pre-deposit* game only if the optimal deposit contract belongs to $(c^{early}, c^{wait}]$. In the next section, we will solve the optimal deposit contract corresponding to each of the three leading cases.

Before we discuss the optimal contract, we give a numerical example in which b and R satisfy condition (5).

Example 1 *The parameters are*

$$a = b = 1.01; p = 0.5; y = 3; R = 1.5.$$

These parameters will be fixed throughout the examples. We see that b and R satisfy (5). Hence c^{early} must be strictly smaller than c^{wait}. We have that $c^{early} = 4.155955$ and $c^{wait} = 4.280878$. Therefore, whenever a contract c is larger than 4.155955 and smaller than or equal to 4.280878, both a run equilibrium and a non-run equilibrium exist in the post-deposit game.

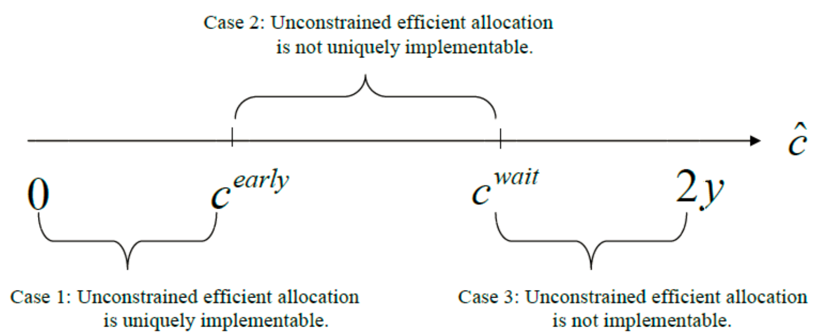
[12] y is not important since it only changes the scale of the economy.

4 The Optimal Contract in the Pre-Deposit Game

When b and R do not satisfy (5), we have $c^{early} \geq c^{wait}$. We have seen that for any $c \in [0, 2y]$ the equilibrium is unique in the *post-deposit* game. Hence the optimal contract c^* in the *pre-deposit* game is the c which maximizes the depositor's welfare at the unique *post-deposit* game equilibrium. Since the equilibrium in the *post-deposit* game is unique, c^* must be smaller than c^{early} and a full bank run is not tolerated.

For the rest of the paper, we focus on the values of b and R such that condition (5) is satisfied. When b and R satisfy (5), for any $c \in (c^{early}, c^{wait}]$ we have multiple equilibria in the *post-deposit* game. We assume – as in PS – that bank runs are *sunspot-driven*. Whether bank runs occur in the *pre-deposit* game depends on whether the optimal contract belongs to the set $(c^{early}, c^{wait}]$. To characterize the optimal contract, we further assume that $a = b$ and divide the problem into three cases depending on $\widehat{c}$, the contract supporting the *unconstrained efficient allocation*. These three cases are: $\widehat{c} \leq c^{early}$ (Case 1), $\widehat{c} \in (c^{early}, c^{wait}]$ (Case 2), and $\widehat{c} > c^{wait}$ (Case 3). We next characterize the parameters for each case. To be more specific, when b and R satisfy (5) and $a = b$, we show that each case corresponds to certain range of the parameter A, the impulse multiplier in the impatient consumer's utility function.

Figure 2. Three Cases



4.1 The Impulse Parameter A and the 3 Cases

The contract $\widehat{c}$ supports the *unconstrained efficient allocation*, where $\widehat{c}$ is defined by

$$\widehat{c} = \arg \max_{c \in [0,2y]} \widehat{W}(c),$$

where

$$\widehat{W}(c) = p^2[u(c)+u(2y-c)]+2p(1-p)[u(c)+v[(2y-c)R]]+2(1-p)^2 v(yR). \quad (6)$$

$\widehat{W}(c)$ is the depositor's expected utility if the types of the depositors are observable ex post.[13] Given the CRRA utility function and the assumption that $a = b$, we have

$$\widehat{c} = \frac{2y}{\{p/(2-p) + 2(1-p)/[(2-p)AR^{b-1}]\}^{1/b} + 1}. \quad (7)$$

From (7), we know that $\widehat{c}$ is an increasing function of A. To reflect this property, we write $\widehat{c}$ as $\widehat{c}(A)$. When the "impulse demand" is stronger, the *unconstrained efficient allocation* allows larger first-period withdrawal. Also from equation (7), we have

$$\lim_{A \to 0} \widehat{c}(A) = 0$$

and

$$\lim_{A \to \infty} \widehat{c}(A) = \frac{2y}{[p/(2-p)]^{1/b} + 1}.$$

From equations (14) and (15), we know neither c^{early} nor c^{wait} depends on A. This is intuitive: c^{early} and c^{wait} are determined by the patient depositor's utility which is independent of A. Hence if A is sufficiently small, we must have Case 1. Furthermore, if

$$\frac{2y}{[p/(2-p)]^{1/b} + 1} \leq c^{early},$$

[13] $\widehat{W}(c)$ is also the depositor's expected utility in the non-run equilibrium of the *post-deposit* game.

only Case 1 obtains.

If

$$c^{early} < \frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{wait},$$

there is a unique level of A, denoted by A_1, such that

$$\widehat{c}(A_1) = c^{early}. \tag{8}$$

Hence if $A \leq A_1$, we have Case 1. And if $A > A_1$, we have Case 2. Case 3 does not exist.

If

$$c^{wait} < \frac{2y}{[p/(2-p)]^{1/b}+1},$$

there exists a unique level of A, denoted by A_2, such that

$$\widehat{c}(A_2) = c^{wait}. \tag{9}$$

We have all three cases now: if $A \leq A_1$, we have Case 1; if $A_1 < A \leq A_2$, we have Case 2; if $A > A_2$, we have Case 3.[14]

Example 2 *We have shown that* $c^{early} = 4.155955$ *and* $c^{wait} = 4.280878$*. Now we calculate the thresholds of A for each case. It is easy to check that* $c^{wait} < \frac{2y}{[p/(2-p)]^{1/b}+1}$ *for the specified parameter values. Therefore, all three cases exist. We have* $A_1 = 6.217686$ *and* $A_2 = 10.277988$*. Hence if* $A \leq 6.217686$*, we are in Case 1; If* $6.217686 < A \leq 10.277988$*, we are in Case 2; If* $A > 10.277988$*, we are in Case 3.*

In Appendix 2, we discuss how other parameters, namely p, R and b, affect $\widehat{c}$. Unlike A, these parameters also affect the values of c^{early} and/or c^{wait} and $\widehat{c}$ is not monotonic in the parameters. The analysis is slightly more complicated in these cases, but once the parameter values are given, we can

[14] It is easy to see that c^{early} does not depend on p and $y < c^{early} < c^{wait} < Ry$. $\lim_{p\to 1} \frac{2y}{[p/(2-p)]^{1/b}+1} = y$ and $\lim_{p\to 0} \frac{2y}{[p/(2-p)]^{1/b}+1} = 2y$. Hence we know that for sufficiently large p, $\frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{early}$. If $R < 2$, $c^{wait} < \frac{2y}{[p/(2-p)]^{1/b}+1}$ for sufficiently small p. For intermediate values of p, we have $c^{early} < \frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{wait}$.

readily compute the values of $\widehat{c}$, c^{early} and c^{wait} to determine which case is applicable.

4.2 The Optimal Contract

In this subsection, we focus on the parameter values of b and R satisfying condition (5). To characterize the optimal contract, we further assume that $a = b$. We will describe the optimal contract c^* for the three cases specified above.

For an arbitrary contract $c \in [0, 2y]$, we have one or two equilibria in the *post-deposit* game depending on whether c belongs to $(c^{early}, c^{wait}]$ or not. As equation (6) shows, the depositor's expected utility in the non-run equilibrium is $\widehat{W}(c)$. Let $W^{run}(c)$ denote the depositor's expected utility in the run equilibrium (if it exists). It is given by

$$\begin{aligned} W^{run}(c) \;&=\; p^2[u(c) + u(2y - c)] + p(1 - p)[u(c) + v(2y - c) + v(c) + u(2y - c)] \\ &\quad +(1 - p)^2[v(c) + v(2y - c)]. \end{aligned} \tag{10}$$

When $c \leq c^{early}$, only the non-run-equilibrium exists and the depositor's ex ante expected utility is simply $\widehat{W}(c)$. When $c^{early} < c \leq c^{wait}$, both the run-equilibrium and the non-run-equilibrium exist in the *post-deposit* game. If the run is *sunspot-driven* and the run probability is s, the depositor's ex-ante expected utility is $(1 - s)\widehat{W}(c) + sW^{run}(c)$. When $c^{wait} < c \leq 2y$, only the run-equilibrium exists and therefore, no consumer would want to deposit in this bank. Hence when we consider the optimal contract, $[0, c^{wait}]$ is the relevant choice interval for c.

Let $c^*(s)$ denote the optimal contract which maximizes the depositor's ex-ante expected utility in the *pre-deposit* game given the run probability s.[15] We have

$$c^*(s) = \arg \max_{c\in[0,c^{wait}]} W(c; s),$$

[15] At $c^*(s; A)$, consumers must weakly prefer depositing to autarky. This is because the deposit contract can always mimic the autarky allocation by setting $c^*(s; A)$ equal to y. Hence the participation constraint is not an issue for $c \in [0, c^{wait}]$

where

$$W(c;s)=\begin{cases}\widehat{W}(c) & \text{if } c\le c^{early}.\\ (1-s)\widehat{W}(c)+sW^{run}(c) & \text{if } c^{early}<c\le c^{wait}.\end{cases} \tag{11}$$

Case 1 *The unconstrained efficient allocation is strongly implementable, i.e.,* $\widehat{c}\le c^{early}$.

Since the *unconstrained efficient allocation* is strongly implementable, it is straightforward to see that the optimal contract for the pre-deposit game supports the *unconstrained efficient allocation*

$$c^*(s)=\widehat{c}$$

and that a bank run does not occur in equilibrium. Other contracts cannot be optimal because they either deliver lower welfare in the non-run equilibrium or, to make things worse, they may also support a run equilibrium.[16] The next is a numerical example for Case 1.

Example 3 *In Example 2, we have seen that as long as* $A\le A_1=6.217686$, *we have Case 1 for this economy. Let* A *be equal to* 1. *We have* $c^*(s)=\widehat{c}=3.004012$ *for any* $s\in[0,1]$. *Since* $c^*(s)<c^{wait}=4.280878$, *a bank run is not an equilibrium.*

As we have discussed in section 4.1, when A is larger than A_2, we have Case 2:

Case 2 *The unconstrained efficient allocation is weakly implementable, i.e.,* $c^{early}<\widehat{c}\le c^{wait}$.

[16] Note that, as have been discussed in the beginning of Section 4, if $c^{wait}\le c^{early}$ bank runs are not equilibrium at the optimal contract in the *pre-deposit* game either. The difference is the following. If $c^{wait}\le c^{early}$, any implementable allocation is strongly implementable. Hence, non-run equilibrium and run equilibrium cannot co-exist. But in Case 1, non-run equilibrium and run equilibrium cannot co-exist for some contracts but they are not optimal.

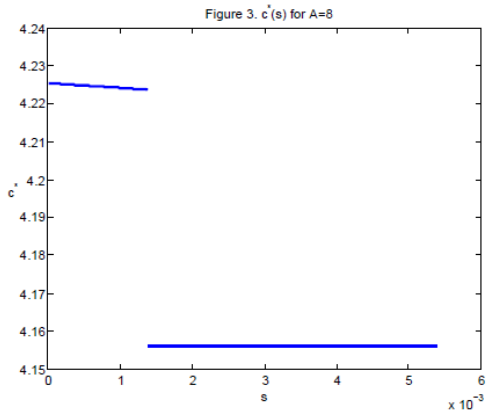
The optimal contract to the pre-deposit game cannot be $\widehat{c}$ except for the degenerate case when $s = 0$. This is because a run equilibrium exists at $\widehat{c}$. How much the optimal contract deviates from $\widehat{c}$ depends on s, which changes the trade-off between the expected utilities over the 2 *post-deposit* game equilibria. We show in Proposition 1 how the optimal contract to the pre-deposit game changes with the probability s.

Proposition 1 *In Case 2, the optimal contract $c^*(s)$ satisfies: (1) If s is larger than the threshold probability s_0 (specified in equation (21) in the proof), the optimal contract is run-proof, $c^*(s) = c^{early}$. (2) If s is smaller than s_0, the optimal contract $c^*(s)$ tolerates runs and it is a strictly decreasing function of s. We have that $c^*(s) \leq \widehat{c}$ (with equality if and only if $s = 0$).*

From Proposition 1, we can see that, in Case 2, the contract supporting the *unconstrained efficient allocation* also supports a run equilibrium in the *post-deposit* game. Except for the degenerate case of the run probability s being zero, that contract cannot be optimal since it delivers very low welfare in the run equilibrium. The optimal contract should optimize the ex-ante trade-off between the depositors' welfares in the run and the non-run equilibria. Intuitively: The trade-off depends on the run probability s. For positive s, a more conservative contract, still tolerating runs or eliminating runs completely, is desirable. If s is larger than the threshold probability s_0, eliminating runs is less costly (in terms of ex ante welfare) than tolerating runs; hence the optimal contract is the best run-proof contract in which the non-run equilibrium is implemented uniquely. On the other hand, if s is smaller than s_0, tolerating runs is less costly. Furthermore, $c^*(s)$ will be strictly decreasing in s since, as s increases, the ex ante welfare leans more towards the welfare in the run equilibrium. The sunspot equilibrium allocation in this case is not a mere randomization over the *unconstrained efficient allocation* and the corresponding run allocation.

Example 4 *In Example 2, we have seen that as long as* $6.217686 < A \leq 10.277988$, *we have Case 2 for this economy. Let A be equal to* 8. *We have that* $s_0 = 0.001382358$. *If $s > s_0$, the optimal contract is run-proof and*

$c^*(s) = c^{early} = 4.155955$. *If* $s < s_0$, *the optimal contract tolerates runs and* $c^*(s)$ *is strictly decreasing in* s *and* $c^*(0) = \widehat{c} = 4.225479$. *If* $s = s_0$, *both the run-proof contract* (c^{early}) *and the run-tolerating contract* ($\widetilde{c}(s_0)$) *are optimal. Figure 3 plots* $c^*(s)$ *of this case.*



As we have discussed in section 4.1, we shift from Case 2 to Case 3 for even larger values of A:

Case 3 *The unconstrained efficient allocation is not implementable, i.e.,* $c^{wait} < \widehat{c}$.

In this case, the *unconstrained efficient allocation* is not implementable since $\widehat{c}$ cannot satisfy the ICC. Like Case 2, the optimal contract for the *pre-deposit* game also involves the trade-off between the two *post-deposit* game equilibria, but the optimal contract also has to maintain the non-run equilibrium (i.e., satisfy the ICC).[17] This changes how $c^*(s)$ tolerates runs. To provide sufficient incentives for the patient depositors to choose late withdrawal, the ICC requires that c be not too large. Hence the binding ICC forces $c^*(s)$ to be more conservative than it would be without the ICC. If the ICC binds, then for small s when runs are still tolerated, c^* is independent of s.

[17] In Case 2, the ICC cannot bind. To see this, firstly we know that a contract c which is larger than the contract supporting the *unconstrained efficient allocation* makes the run equilibrium more devastating. Secondly, it does not improve welfare at the non-run equilibrium. Hence to find the optimal contract, we need only to focus on contracts which are more conservative than the contract supporting the *unconstrained efficient allocation.* So binding incentive compatibility cannot occur in Case 2.

Proposition 2 *In Case 3: (1) If* s *is larger than the threshold probability* s_1 *(specified in equation (26) in the proof), we have* $c^*(s) = c^{early}$ *and the optimal contract is run-proof. (2) If* s *is smaller than* s_1*, the optimal contract* $c^*(s)$ *tolerates runs and it is a weakly decreasing function of* s*. Furthermore, we have* $c^*(s) = c^{wait}$ *for at least part of the run tolerating range of* s*.*

At least when s is sufficiently small, maintaining the non-run equilibrium is dominant in the determination of the optimal contract. So for Case 3, when s is sufficiently small, $c^*(s)$ does not change with s since the ICC does not depend on s. The ICC may bind for just part of the run-tolerating range of s (the first sub-case) or the whole run-tolerating range of s (the second sub-case).[18] When the ICC binds, the allocation supported by the optimal contract is a randomization over the *constrained efficient allocation*[19] and the corresponding run allocation. The *unconstrained efficient allocation* is never supported by $c^*(s)$ since it is not implementable.

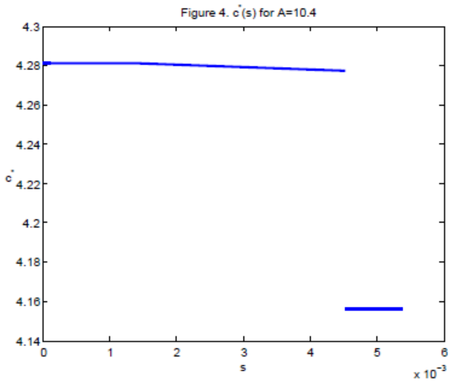
We next provide two examples of the optimal contract for Case 3, which correspond to the two sub-cases.

Example 5 *In Example 2, we have seen that as long as* $A > 10.277988$*, we have Case 3. Let* A *be equal to* 10.4*. We have that* $s_2 = 0.001719643$ *and* $s_4 = 0.004520095$*. Since* $s_4 > s_2$*, we are in the first sub-case. We have that* $s_3 = 0.004524181$*. Hence when* $s < s_2 = 0.001719643$*, the optimal contract tolerates runs and the ICC binds:* $c^*(s) = c^{wait} = 4.280878$*. When* $s_2 \leq s < s_3 = 0.004524181$*, the optimal contract tolerates runs and the ICC does not bind. Hence* $c^*(s) = \overline{c}(s)$ *and it is strictly decreasing in* s*. When* $s = s_3$*, both the run-proof contract* c^{early} *and the run-tolerating contract* $(\overline{c}(s_3))$ *are optimal. When* $s_3 < s$*, the optimal contract is run-proof and* $c^*(s) = c^{early} = 4.155955$*.*

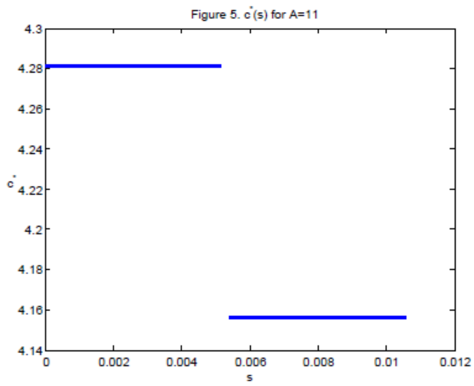
---

[18] The PS example fits in the second sub-case.

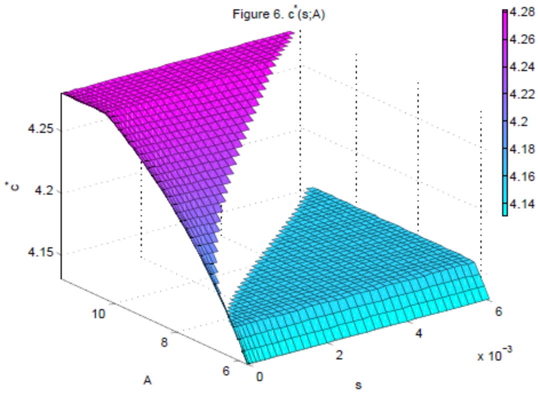
[19] Ennis and Keister (2010) define the *constrained efficient allocation* as “a (contingent) consumption allocation to maximize the ex ante expected utility of agents subject to incentive compatibility, sequential service, and resource feasibility constraints”. We use the same definition.



Example 6 *Let A be equal to* 11. *We have* $s_2 = 0.009591026$ *and* $s_4 = 0.005281242$. *Since* $s_4 \le s_2$, *we are in the second sub-case. Hence when* $s < s_4$, *the optimal contract tolerates runs and the ICC binds.* $c^*(s) = c^{wait} = 4.280878$; *When* $s > s_4$, *the optimal contract is run-proof and* $c^*(s) = c^{early} = 4.155955$; *When* $s = s_4$, *both the run-proof contract* (c^{early}) *and the run-tolerating contract* (c^{wait}) *are optimal.*



We plot c^* versus s and A in Figure 6. If $A \le A_1 = 6.217686$, we are in Case 1 and the optimal contract doesn't tolerate runs and it is equal to $\widehat{c}$ the value of which depends on A. If $6.217686 < A \le 10.277988$, we are in Case 2 and the optimal contract is strictly decreasing in s until it levels off at the best run-proof contract $c^{early} = 4.155955$. If $10.277988 < A$, we are in Case 3 and the ICC binds when s is small. The ICC may bind either in part of the run-tolerating range of s (the first sub-case) or the whole run-tolerating range of s (the second sub-case).



In Figure 7, we plot the welfare loss (measured in percentage of endowment) from being restricted to using for small s the contract supporting the constrained efficient allocation and for large s to the best run-proof contract, instead of using $c^*(s)$. In our calculations, we assume the bank is able to choose the jump probability freely. If the jump probability is forced to be the same as in $c^*(s)$, the welfare loss would typically be greater than that shown in Figure 7. Since $c^*(s)$ equals the best run-proof contract for large s, there is no welfare loss for large s. If we are in Case 1, the welfare loss is 0 since $c^*(s)$ is the same as the contract supporting the constrained efficient allocation. If we are in Case 2, the welfare loss is positive whenever runs are tolerated by $c^*(s)$ and the loss is larger as the run probability increases. This is because $c^*(s)$ is different from the contract supporting the constrained efficient allocation (except when $s = 0$) and, as s increases, $c^*(s)$ is smaller but the contract supporting the constrained efficient allocation (for fixed A) doesn't change. If we are in Case 3, the welfare loss is 0 for small s for which ICC binds, which makes $c^*(s)$ the same as the contract supporting the constrained efficient allocation.

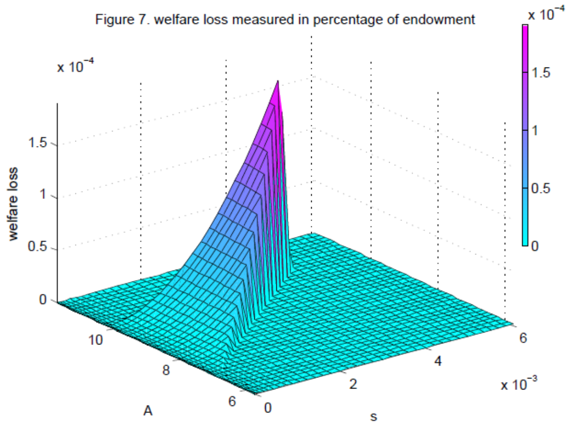


Figure 7. welfare loss measured in percentage of endowment

5 Summary

PS show that bank runs (driven by sunspots) can be equilibria in the *pre-deposit* game of the corresponding DD-type *post-deposit* banking model. The optimal contract to the *pre-deposit* game in the PS example is a step-function of the run-probability: the optimal contract tolerating runs does not change with the run-probability until the probability reaches the threshold at which the optimal contract switches to the best run-proof contract.

In this paper, the general form of the optimal contract to the *pre-deposit* game is analyzed for different parameters. We first characterize the parameters such that multiple equilibria, non-run and run, coexist in the *post-deposit* game. When multiple equilibria coexist, the *unconstrained efficient allocation* falls into one of the three cases: (1) it is strongly implementable, (2) it is weakly implementable, or (3) it is not implementable. We characterize the parameters for each case. Bank runs matter in the last two cases. In both of these cases, the optimal contract switches to being run-proof when the run probability is sufficiently large. When runs are tolerated, whether the optimal contract should be more conservative for a larger run probability differs in the two cases: In Case 2, the ICC doesn't bind since the *unconstrained efficient allocation* is (weakly) implementable. As a result of balancing the trade-off between the run equilibrium and non-run equilibrium in the *post-deposit* game, the optimal contract adjusts continuously and becomes more conservative as the run probability increases. However, in Case 3, the ICC binds for small run-probabilities, which forces the contract to be more conservative than it would have been without the constraint. Hence, for Case 3, the optimal contract does not change with s until the ICC no longer binds.

The implication of identifying the cases of the optimal contract is that *how* bank runs are tolerated can be complicated. As the economy's level of fragility or pessimism (indicated by the probability s) changes, how the banking contract and bank regulation should respond is different for different cases.

Our paper makes a contribution to the wider literature on sunspot equilibrium. This is yet another example in which not all sunspot equilibria are mere randomizations over certainty equilibria. See Shell (2008).

6 Appendix 1

6.1 Proof of Lemma 1

Proof. Inequality (3) holds if and only if

$$\frac{-(c^{1-b})/2 + (2y-c)^{1-b}(R^{1-b} - 1/2)}{(b-1)} > 0.$$

For $c \in [0, 2y]$ to satisfy the above inequality, we must have $(R^{1-b} - 1/2) > 0$, which can be re-written as

$$b < 1 + \ln 2/\ln R. \tag{12}$$

When b and R satisfy condition (12), inequality (3) is equivalent to

$$c \in (c^{early}, 2y], \tag{13}$$

where

$$c^{early} = 2y/[(2/R^{b-1} - 1)^{1/(b-1)} + 1]. \tag{14}$$

That is, c^{early} is defined to be the level of c *beyond* which a patient depositor chooses early withdrawal if he expects that the other patient depositor will also choose early withdrawal. The superscript "early" stands for the patient depositor's expectation that the other patient depositor will choose early withdrawal. Hence the set of c satisfying condition (3) is non-empty if and only if b and R satisfy inequality (12).

Now, we move to inequality (4). The difference between the left-hand side and the right hand side of inequality (4) is a continuous function of c. When inequality (12) holds, the difference is decreasing in c. It changes from $+\infty$ when $c = 0$ to $-\infty$ when $c = 2y$. Hence there is a unique level of $c \in (0, 2y)$, such that (4) holds with equality. Denote that level of c by c^{wait}. That is, c^{wait} is defined by

$$pv[(2y - c^{wait})R] + (1-p)v(yR) = p[v(c^{wait}) + v(2y - c^{wait})]/2 + (1-p)v(c^{wait}). \tag{15}$$

Thus c^{wait} is the level of c *below* which a patient depositor chooses late withdrawal if he expects that the other patient depositor will also choose late withdrawal. The superscript "wait" stands for the patient depositor's expectation that the other patient depositor will wait and choose late withdrawal. So when b and R satisfy condition (12), inequality (4) is equivalent to

$$c \in [0, c^{wait}]. \tag{16}$$

For c to satisfy both condition (3) and condition (4), we also need

$$c^{wait} > c^{early}. \tag{17}$$

To get the condition on b and R for inequality (17) to hold, we merely need to replace c in inequality (4) by c^{early} and require that inequality (4) holds. This results in

$$\frac{2/R}{(2/R^{b-1} - 1)^{1/(b-1)} + 1} < 1. \tag{18}$$

When b and R satisfy condition (12), $(2/R^{b-1} - 1)^{1/(b-1)}$ is decreasing in b.

Hence inequality (18) is equivalent to

$$b < 2 \tag{19}$$

Therefore, given inequality (12), the set of c satisfying condition (4) is non-empty if and only if b is smaller than 2. To summarize, the set of c satisfying both condition (3) and (4) is non-empty if and only if b and R satisfy both inequality (12) and inequality (19), which results in condition (5). ■

6.2 Proof of Proposition 1

Proof. Since $\widehat{W}(c) > W^{run}(c)$, $W(c;s)$ is not continuous at c^{early} if $s > 0$. We study the two regions $[0, c^{early}]$ and $(c^{early}, c^{wait}]$ separately, and compare the maximum values of $W(c;s)$ in these two regions.

For $c \in [0, c^{early}]$, $W(c;s)$ is strictly increasing in c since $c^{early} < \widehat{c}$. Hence the maximum value of $W(c;s)$ over $[0, c^{early}]$ is achieved at c^{early}. Therefore the best run-proof contract is $c = c^{early}$.

For $c \in (c^{early}, c^{wait}]$, the maximum value of $W(c;s)$ may not be achievable because $(c^{early}, c^{wait}]$ is not closed. To fix this problem, we define a function $\widetilde{W}(c;s)$ on $[c^{early}, c^{wait}]$ by

$$\widetilde{W}(c;s) = (1-s)\widehat{W}(c) + sW^{run}(c).$$

When $c \in (c^{early}, c^{wait}]$, $\widetilde{W}(c;s) = W(c;s)$. When $c = c^{early}$, $\widetilde{W}(c;s) < W(c;s)$. Let $\widetilde{c}(s)$ be defined by

$$\widetilde{c}(s) = \arg \max_{c \in [c^{early}, c^{wait}]} \widetilde{W}(c;s).$$

We have

$$\widetilde{c}(s) = \max\{\frac{2y}{\gamma^{1/b}+1}, c^{early}\}, \tag{20}$$

where

$$\gamma = \frac{s(1-p)(pA + 1 - p\frac{2}{R^{a-1}}) + (p^2A + (1-p)p\frac{2}{R^{a-1}})}{s(1-p)(1-pA) + p(2-p)A}.$$

It can be shown that $\widetilde{c}(s)$ is continuous in s. Furthermore, $\widetilde{c}(s)$ is strictly decreasing in s when s is small such that $\widetilde{c}(s) > c^{early}$.[20] We also have $c^{early} = \widetilde{c}(1) < \widetilde{c}(0) = \widehat{c}$. $\widetilde{W}(\widetilde{c}(s); s)$ is continuous in s and it is also strictly decreasing in s since $\widehat{W}(c) > W^{run}(c)$. Furthermore, we have

$$\widetilde{W}(\widetilde{c}(0); 0) = \widehat{W}(\widehat{c}) > \widehat{W}(c^{early})$$

and

$$\widetilde{W}(\widetilde{c}(1); 1) = W^{run}(c^{early}) < \widehat{W}(c^{early}).$$

Hence there is a unique $s_0 \in (0, 1)$ such that

$$\widetilde{W}(\widetilde{c}(s_0); s_0) = \widehat{W}(c^{early}). \tag{21}$$

Obviously, $\widetilde{c}(s_0) > c^{early}$.

Hence if $s < s_0$, we have $c^*(s) = \widetilde{c}(s)$. The optimal contract $c^*(s)$ tolerates runs and it is a strictly decreasing function of s. We have $c^{early} < c^*(s) \leq \widehat{c}$ (with equality if and only if $s = 0$).

If $s > s_0$, $c^*(s) = c^{early}$. The optimal contract is run-proof.

If $s = s_0$, $\widetilde{W}(\widetilde{c}(s); s) = \widehat{W}(c^{early})$. So both the run-proof contract (c^{early}) and the run-tolerating contract $(\widetilde{c}(s_0))$ are optimal. ■

6.3 Proof of Proposition 2

Proof. The proof is similar to that for Proposition 1. The only difference is that the ICC may bind. We still study the two regions $[0, c^{early}]$ and $(c^{early}, c^{wait}]$ separately, and compare the maximum values of $W(c; s)$ in these two regions.

For $c \in [0, c^{early}]$, it is easy to see that $W(c; s)$ is strictly increasing. Hence, as in Case 2, the best run-proof contract is still $c = c^{early}$.

For $c \in (c^{early}, c^{wait}]$, the maximum value of $W(c; s)$ may not be achievable because $(c^{early}, c^{wait}]$ is not closed. To fix that problem and characterize the

[20] It is easy to check that if $\widetilde{c}(s; A) > c^{early}$. $\widetilde{c}(s; A)$ is strictly decreasing in s because $AR^{a-1} > 1$. $AR^{a-1} > 1$ must hold in Case 2. To see why, it is trivial to establish that c^{early} must be larger than y. Hence in Case 2, we have $\widehat{c}(A) > y$, which leads to $AR^{a-1} > 1$.

possibly binding ICC, we define a function $\overline{W}(c;s)$ on $[c^{early}, 2y]$:

$$\overline{W}(c;s) = (1-s)\widehat{W}(c) + sW^{run}(c).$$

When $c \in (c^{early}, c^{wait}]$, $\overline{W}(c;s) = W(c;s)$. When $c = c^{early}$, $\overline{W}(c;s) < W(c;s)$. Let $\overline{c}(s)$ be defined by

$$\overline{c}(s) = \arg \max_{c \in [c^{early}, 2y]} \overline{W}(c;s).$$

We have that

$$\overline{c}(s) = \frac{2y}{\eta^{1/b}+1}, \tag{22}$$

where

$$\eta = \frac{s(1-p)(pA+1-p\frac{2}{R^{a-1}}) + (p^2 A + (1-p)p\frac{2}{R^{a-1}})}{s(1-p)(1-pA)+p(2-p)A}.$$

By using the same argument as that in Proposition 2, we can show that $\overline{c}(s)$ is continuous in s. Furthermore, $\overline{c}(s)$ is strictly decreasing in s when s is small such that $\overline{c}(s) > c^{early}$. We also have that $c^{early} = \overline{c}(1) < \overline{c}(0) = \widehat{c}$. Note that in Case 3, $c^{wait} < \widehat{c}$. Hence there is a unique level of $s \in (0,1)$, denoted by s_2, such that

$$\overline{c}(s_2) = c^{wait}. \tag{23}$$

That is, s_2 is the threshold run probability below which the ICC binds. Next, we need to check, when $s = s_2$, whether the optimal contract still tolerates runs. To do that, we define s_4 by

$$s_4 = \frac{\widehat{W}(c^{wait}) - \widehat{W}(c^{early})}{\widehat{W}(c^{wait}) - W^{run}(c^{early})}. \tag{24}$$

Obviously, $s_4 \in (0,1)$. There will be two sub-cases depending on whether the optimal contract still tolerates runs when when $s = s_2$.

In the first sub-case of Case 3, $s_4 > s_2$, that is, at the threshold run probability which makes the ICC just become non-binding, the optimal contract still tolerates runs. Now we need to determine the threshold run probability beyond which the optimal contract switches to being run-proof. That threshold level is s_3 which is defined by

$$\overline{W}(\overline{c}(s_3); s_3) = \widehat{W}(c^{early}). \tag{25}$$

By using the same argument as that in Proposition 2, we know that $\overline{W}(\overline{c}(s); s)$ is continuous and strictly decreasing in s. Therefore, s_3 is unique and it is well defined. Since $s_4 > s_2$, we know that $s_3 > s_2$. $c^*(s)$ satisfies the following property: When $s < s_2$, the ICC binds and $c^*(s) = c^{wait}$ since

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) > \widehat{W}(c^{early}).$$

When $s_2 \leq s < s_3$, the ICC no longer binds and $c^*(s) = \overline{c}(s)$ since

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) > \widehat{W}(c^{early}).$$

When $s = s_3$, both $\overline{c}(s)$ and c^{early} are optimal since

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) = \widehat{W}(c^{early}).$$

When $s > s_3$, $c^*(s) = c^{early}$ since

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) < \widehat{W}(c^{early}).$$

To summarize, if $s_4 > s_2$ we have

$$c^*(s) = \begin{cases} c^{wait} \text{ if } s < s_2 \\ \overline{c}(s) \text{ if } s_2 \leq s \leq s_3 \\ c^{early} \text{ if } s_3 \leq s. \end{cases}$$

In the second sub-case of Case 3, $s_4 \leq s_2$, that is, at the run probability which makes the ICC just become non-binding, the optimal contract does not tolerate runs. Hence the optimal contract will switch to the best run-proof contract (c^{early}) when the ICC still binds. $c^*(s)$ satisfies the following property: When $s < s_4$, the ICC binds and $c^*(s) = c^{wait}$ since

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) > \widehat{W}(c^{early}).$$

When $s = s_4$, both c^{wait} or c^{early} are optimal since

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s_4) = \widehat{W}(c^{early}).$$

When $s_4 < s < s_2$, $c^*(s) = c^{early}$. This is because the ICC binds and

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) < \widehat{W}(c^{early}).$$

When $s_2 \leq s$, $c^*(s; A)$ is still equal to c^{early}. This is because the ICC no longer binds and

$$W(\bar{c}(s); s) = \overline{W}(\bar{c}(s); s) < \overline{W}(\bar{c}(s_2); s_2) = \overline{W}(c^{wait}; s_2) < \widehat{W}(c^{early}).$$

To summarize, if $s_4 \leq s_2$, we have

$$c^*(s) = \begin{cases} c^{wait} \text{ if } s \leq s_4 \\ c^{early} \text{ if } s \geq s_4. \end{cases}$$

We can see, in both of the two sub-cases, $c^*(s)$ switches to run-proof if the run probability is larger than the threshold. Let s_1 denote that threshold run probability and we can have

$$s_1 = \begin{cases} s_3 \text{ if } s_4 > s_2 \\ s_4 \text{ if } s_4 \leq s_2. \end{cases} \tag{26}$$

■

7 Appendix 2

In section 4.1, we have shown how different values of A correspond to the three cases of $\widehat{c}$, which determines *how* the optimal contract tolerates runs.

In this section, we will discuss how other parameters, namely p, R and b, affect $\widehat{c}$. We limit our discussion to the set of parameters permiting strategic complementarity, i.e., b and R satisfying inequality (5).

7.1 Probability of impatience p

From equation (7), it is easy to show that $\widehat{c}$ is increasing in p if $AR^{b-1} < 1$. $\widehat{c}$ is equal to y if $AR^{b-1} = 1$. And $\widehat{c}$ is decreasing in p if $AR^{b-1} > 1$. Hence how p affects $\widehat{c}$ depends on the values of A and R. The intuition is the following. Because there is aggregate uncertainty, the economy may have 2 impatient consumers, 1 impatient consumer and 1 patient consumer, or 2 patient consumers. p changes the likelihood ratio between the first and the second scenarios.[21] The first scenario requires no cross-subsidy between the consumers. The second scenario requires cross-subsidy, but how it is conducted depends on A and R. If $AR^{b-1} < 1$, the subsidy is from the impatient to the patient (i.e., $\widehat{c} < y$). While if $AR^{b-1} > 1$, the subsidy is from the patient to the impatient (i.e., $\widehat{c} > y$). When p increases, the second scenario becomes less likely compared to the first one and less subsidy needs to be undertaken (i.e., $\widehat{c}$ should be closer to y). Hence if $AR^{b-1} < 1$, $\widehat{c}$ increases as p increases. And if $AR^{b-1} > 1$, the opposite is true.

To see how different values of p correspond to the three cases of the optimal contract, note that c^{early} doesn't depend on p and

$$\lim_{p \to 1} \widehat{c} = y < c^{early}.$$

Hence we are in Case 1 whenever p is sufficiently large. Furthermore,

$$\lim_{p \to 0} \widehat{c} = \frac{2y}{(1/AR^{b-1})^{1/b} + 1}$$

Hence if $\frac{2y}{(1/AR^{b-1})^{1/b}+1} \leq c^{early}$, only Case 1 obtains.

If

$$\frac{2y}{(1/AR^{b-1})^{1/b} + 1} > c^{early},$$

[21] The last scenario doesn't matter since the welfare there won't be affected by c.

which implies $AR^{b-1} > 1$, there is a unique level of p, denoted by p_1, such that

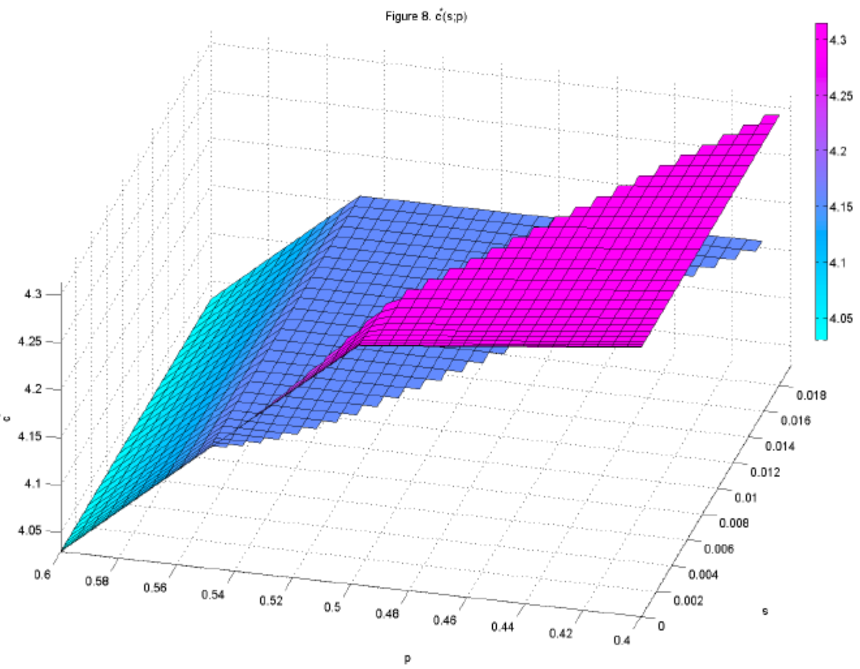
$$\widehat{c}(p_1) = c^{early}.$$

If $p \geq p_1$, we are in Case 1. If $p < p_1$, we are in Case 2 or Case 3 depending on whether $\widehat{c}(p)$ is smaller than than c^{wait} or not. Note that c^{wait} changes with p.[22]

Example 7 *Let*

$$a = b = 1.01; A = 10; y = 3; R = 1.5.$$

We have that $c^{early} = 4.155955$. *It is easy to check that if* $p \geq 0.548823$, *we are in Case 1. If* $0.497423 \leq p < 0.548823$, *we are in Case 2. If* $p < 0.497423$, *we are in Case 3.*



[22] In fact, c^{wait} is decreasing in p. This is quite intuitive. When a patient depositor makes his withdrawal decision, larger p implies that he expects that it is more likely that the other depositor is impatient and, therefore, withdraws early. Hence it is harder to make the patient depositor withdraw late.

We plot c^* versus s and p in Figure 8. If $p \geq 0.548823$, we are in Case 1 and the optimal contract doesn't tolerate runs and it is equal to $\widehat{c}$ the value of which depends on p. If $0.497423 \leq p < 0.548823$, we are in Case 2 and the optimal contract is strictly decreasing in s until it levels off at the best run-proof contract $c^{early} = 4.155955$. If $p < 0.497423$, we are in Case 3 and the ICC binds when s is small.

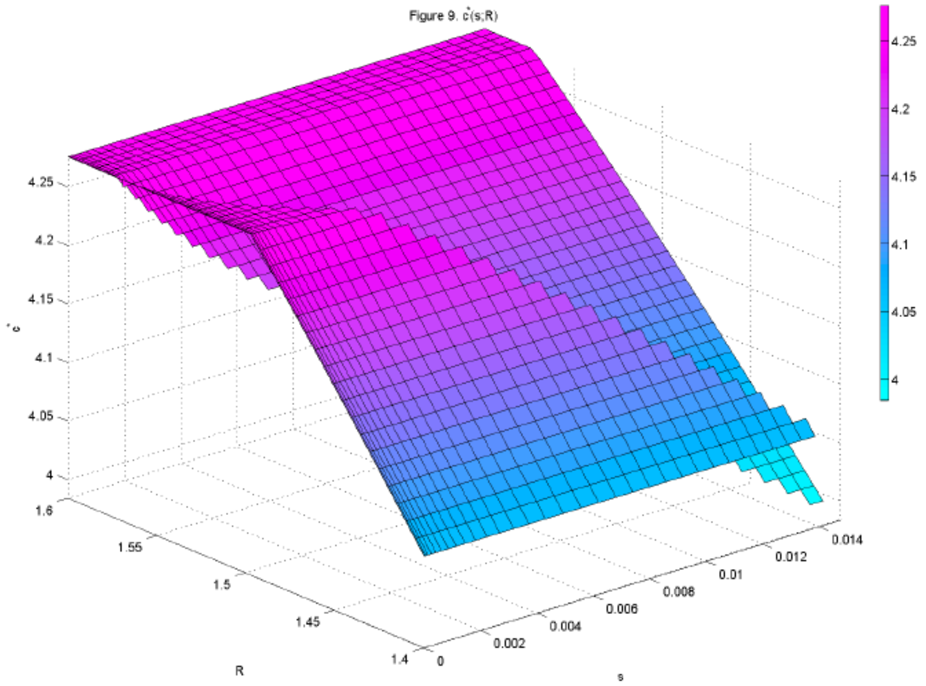
7.2 Return on bank investment R

From equation (7), it is easy to show that $\widehat{c}$ is increasing in R. R affects $\widehat{c}$ by changing the optimal allocation when the economy has one impatient depositor and one patient depositor. For larger R, on the one hand, the marginal rate of transformation between the first period consumption and the second period consumption is higher. On the other hand, the marginal rate of substitution between the first period consumption by the impatient depositor for the second period consumption by the patient depositor is also higher. Since $b > 1$, the second effect is stronger and, therefore, the optimal allocation allows more first-period withdrawal, i.e., $\widehat{c}$ increases as R increases. It is easy to see that both c^{early} and c^{wait} increase in R. If $\widehat{c} \leq c^{early}$, we are in Case 1. If $c^{early} < \widehat{c} \leq c^{wait}$, we are in Case 2. If $\widehat{c} > c^{wait}$, we are in Case 3.

Example 8 *Let*

$$a = b = 1.01; A = 10; y = 3; p = 0.5.$$

It is easy to check that if $R \geq 1.572948$*, we are in Case 1. If* $1.497374 \leq R < 1.572948$*, we are in Case 2. If* $R < 1.497374$*, we are in Case 3.*



We plot c^* versus s and R in Figure 9. If $R \geq 1.572948$, we are in Case 1 and the optimal contract doesn't tolerate runs and it is equal to $\widehat{c}$ the value of which depends on R. If $1.497374 \leq R < 1.572948$, we are in Case 2 and the optimal contract is strictly decreasing in s until it levels off at the best run-proof contract c^{early}. Note that c^{early} increases in R. If $R < 1.497374$, we are in Case 3 and the ICC binds when s is small.

7.3 Risk aversion parameter b

To make the analysis consistent with other comparative statics, let $a = b$. The sign of $\frac{\partial \widehat{c}}{\partial b}$ is the same as the sign of

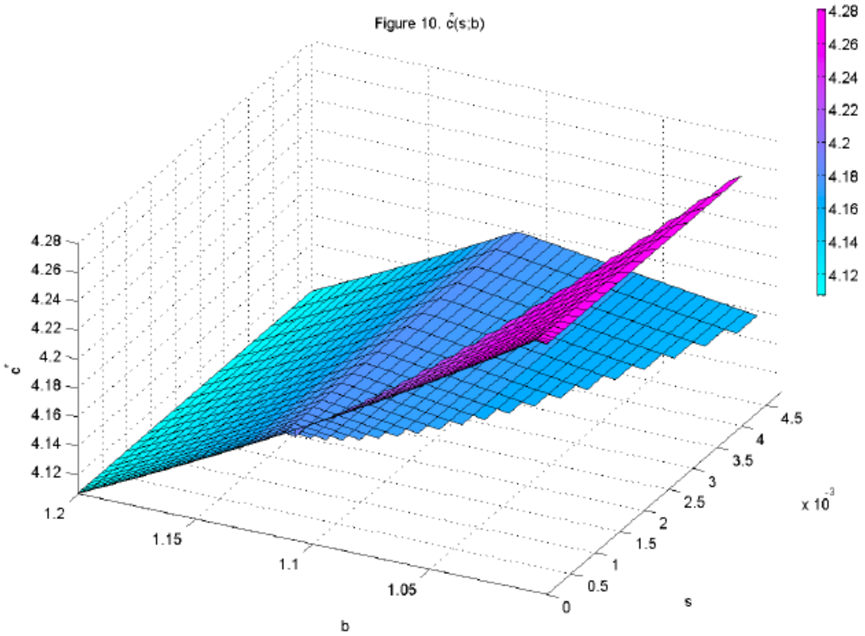
$$\ln\left(\frac{p}{2-p} + \frac{2(1-p)}{(2-p)AR^{b-1}}\right) + \frac{2(1-p)b\ln(R)}{2(1-p) + pAR^{b-1}}.$$

Hence if A is smaller than a threshold level, $\frac{\partial \widehat{c}}{\partial b} > 0$. Otherwise, we have $\frac{\partial \widehat{c}}{\partial b} < 0$. The intuition is the following. As b increases, consumption smoothing across the two depositors is more desirable. When A is small, $\widehat{c}$ is small and more consumption smoothing requires larger $\widehat{c}$. When A is large, $\widehat{c}$ is large and more consumption smoothing requires smaller $\widehat{c}$.

Example 9 *Let*

$$A = 10; y = 3; p = 0.5; R = 1.5.$$

It is easy to check that if $b \geq 1.112528$*, we are in Case 1. If* $1.00524 \leq b < 1.112528$*, we are in Case 2. If* $b < 1.00524$*, we are in Case 3.*



We plot c^* versus s and b in Figure 10. If $b \geq 1.112528$, we are in Case 1 and the optimal contract doesn't tolerate runs and it is equal to $\widehat{c}$ the value of which depends on b. If $1.00524 \leq b < 1.112528$, we are in Case 2 and the optimal contract is strictly decreasing in s until it levels off at the best run-proof contract c^{early}. If $b < 1.00524$, we are in Case 3 and the ICC binds when s is small.

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