Economics 6130-2
Macroeconomics I, Part 2
Solutions to Practice Questions for Final

1 Overlapping Generations, I

2 period lives.
1 commodity per period, $l = 1$.
Stationary endowments:

$$\omega_0^1 = 2 > 0 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (3, 2) > 0 \text{ for } t = 1, 2, ...$$

Stationary preferences:

$$u_0(x_0^1) = 10 \log x_0^1 \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = \log x_t^t + 10 \log x_t^{t+1} \text{ for } t = 1, 2, ...$$

Money taxation-transfer:

$$m_1^1 = 50,\ m_1^2 = 100, \qquad m_t^s = 0 \text{ otherwise}$$

Goods price of money is $p^m \geq 0$.

Derive the translated, reflected offer curve.

Analyze the global dynamics. Be precise. Find steady-state equilibria. Describe all possible paths. Include in your answer: hyperinflation, hyperdeflation, bursting bubbles, non-bursting bubbles.

Solution:

The reflected offer curve is given by

$$z^{t+1} = \frac{CBs^t}{DA - (D+C)s^t},$$

for generic OLG economies with stationary endowments

$$\omega_0^1 = B > 0 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (A, B) > 0 \text{ for } t = 1, 2, ...$$

and stationary log preferences

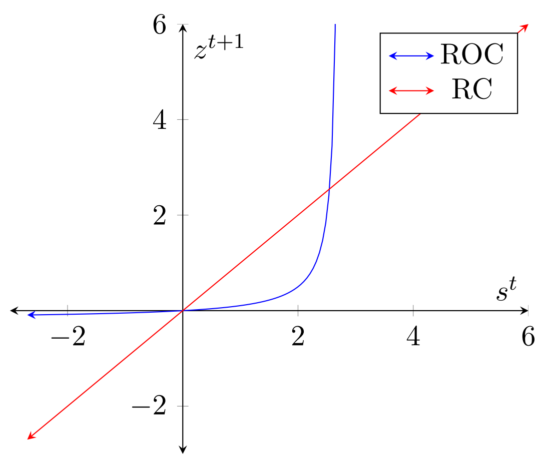
$$u_0(x_0^1) = D\log(x_0^1) \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = C\log(x_t^t) + D\log(x_t^{t+1}) \text{ for } t = 1, 2, ...$$

See solution to Problem Set 3, Question 1. **Note:** Citing a result from a problem set solution is unacceptable for your work on an examination. Show your work!

...Thus the reflected offer curve for this problem is given by

$$z^{t+1} = \frac{2s^t}{30 - (11)s^t},$$

Graphically:



The global dynamics we get are that of Samuelson. The kicker is that the initial old born at date-0 have no money endowment to exchange for the date-1 generation's desired excess supply of $s^1 = 28/11$. Thus this monetary economy is stuck in autarky at date-1, and the date-1 generation is forced to save their first period money endowment $m_1^1 = 50$. Thus in the second period, the date-1 generation has $m_1^1 + m_1^2 = 150$ units of money to exchange with the date-2 generation to finance excess demand of $z^2 = 28/11$. Thus there are 2 stationary states.

(a) The non-monetary (non-PO) steady state with $s^t = z^{t+1} = 0$ (autarky).

(b) The monetary (PO) steady state with $s^t = z^{t+1} = 150\bar{p}^m = \frac{28}{11}$ and $\bar{p}^m = \frac{28}{11} \times \frac{1}{150} = \frac{14}{825}$.

The non-monetary steady state is locally unstable at date-1, as the economy will jump to the monetary equilibrium at date-2, and is stable thereafter. The monetary steady state is always unstable.

If $0 < p^m < \bar{p}^m$, the economy is inflationary. The current goods price of money tends asymptotically to zero. The money bubble fades away, but it does not burst.

If $p^m > \bar{p}^m$, the economy is deflationary. The current goods price of money grows so that in finite time demand for goods exceeds supply, so the deflationary bubble must burst (in finite time) since $x_t^t + x_t^{t+1} > \omega_t^t + \omega_t^{t+1}$, i.e. goods market clearing fails. The hyper-deflationary path does not satisfy long-run perfect foresight. The bubble must burst in finite time.

2 Heterogeneous Capital

$$Y = C + Z_1 + 4Z_2 = 5K^{\frac{1}{3}}L^{\frac{2}{3}}$$
$$K_1 + 2K_2 = K$$
$$\dot{K}_1 = Z_1$$
$$\dot{K}_2 = Z_2$$
$$C = \frac{95Y}{100}$$
$$\dot{L} = \frac{L}{50}$$

Let p_i be the current price of a type i machines in terms of consumption, $i = 1, 2$.

2a. What restrictions on prices are imposed by momentary (i.e., static) equilibrium?

2b. What are the costs in terms of output Y of newly produced capital good 1 and capital good 2? What are the relative values of old machines of type 1 and type 2? Relate this to the costs of replacement.

2c. Let $\dot{p}_i^e$ be the expected change in p_i. What is the asset market clearing equation?

2d. Do the full dynamic analysis for the case of static expectations, i.e. $\dot{p}_i^e = 0$ for $i = 1, 2$.

2e. Do the full dynamic analysis for the case of short-run perfect foresight, i.e. $\dot{p}_i^e = \dot{p}_i$ for $i = 1, 2$.

2f. How do the dynamics for this heterogeneous capital model compare with those for the OG pure-exchange (or endowment) economy?

Solution:

Overview from the planner's viewpoint: Since $0 < C < Y$, the consumption good and at least one capital good must be produced. Let $p_c = 1$. Investment good 2 is 4 times as expensive to produce but only 2 times as productive. Hence $Z_2 = 0$ and $Z_1 > 0$. No new machines of type 2 should be produced, but there might be old machines left over from a previous era. Since type 2 old machines are twice as productive, we suggest $p_1 = p_c = 1$ and $p_2 = 2$, yielding $Z_1 > 0$ and $Z_2 = 0$.

2a: $\max(p_1, \frac{p_2}{4}) = 1$ and $Z_2 = 0$

2b: The cost of newly produced capital is 1 unit of output for a unit of capital good 1, and 4 units of output for a unit of capital good 2. The relative value of old machines of capital good 2 is twice that of old machines of capital good 1, because the marginal product of K_2 is twice that of K_1.

2c:

$$\frac{\dot{p_1}^e}{p_1} + \frac{f_1}{p_1} = \frac{\dot{p_2}^e}{p_2} + \frac{f_2}{p_2}$$
$$\frac{\dot{p_1}^e}{p_1} + \frac{f_1}{p_1} = \frac{\dot{p_2}^e}{p_2} + \frac{2f_1}{p_2}$$
$$k = k_1 + 2k_2$$
$$f(k) = f(k_1 + 2k_2)$$
$$f_1 = \frac{5}{3}k^{-\frac{2}{3}}$$
$$f_2 = \frac{10}{3}k^{-\frac{2}{3}} = 2f_1$$

2d:

$$\dot{p_1}^e = \dot{p_2}^e = 0$$
$$p_1 = 1$$
$$f_1 = \frac{2f_1}{p_2}$$
$$p_2 = 2$$

In this case, static expectations leads to the planner solution with:

$$\lim_{t\to\infty} k_1(t) = (\frac{5}{100} \times 5 \times 50)^{\frac{3}{2}} = (\frac{25}{2})^{\frac{3}{2}}$$
$$\lim_{t\to\infty} k_2(t) = 0$$

2e:

$$\frac{\dot{p_1}}{p_1} + \frac{f_1}{p_1} = \frac{\dot{p_2}}{p_2} + \frac{2f_1}{p_2}$$

Assume $p_1 = 1$ and hence $\dot{p_1} = 0$

$$f_1 = \frac{\dot{p_2}}{p_2} + \frac{2f_1}{p_2}$$
$$\frac{\dot{p_2}}{p_2} = f_1(1 - \frac{2}{p_2})$$

If $p_2 = 2$, $\dot{p_2} = 0$. So $p_1 = 1, p_2 = 2, Z_2 = 0$ yields the planner solution with:

$$\lim_{t\to\infty} k_1(t) = (\frac{5}{100} \times 5 \times 50)^{\frac{3}{2}} = (\frac{25}{2})^{\frac{3}{2}}$$
$$\lim_{t\to\infty} k_2(t) = 0$$

Next assume (for example) that $2 < p_2 < 4$.

$$\dot{p_2} = p_2 f_1(1 - \frac{2}{p_2}) > 0$$

f_2 bounded from below
p_2 increasing faster than absolute constant rate.
$p_2 > 4$ in finite time producers will produce only 2nd capital good but $C = 0$. Disequilibrium. Bubble must burst if $p_2 < 1$.

2f: Compare the above dynamics with the answer to Question 1. In the OG pure-exchange economy, the hyper-deflation bubble must burst because eventually in finite time the current consumption value of money exceeds the aggregate endowment. In the growth model, the hyper-deflation bubble must burst because in finite time, the price of used capital is below replacement cost leading in finite time to the price of capital zero even though its net marginal product is positive.

3 Overlapping Generations, II

You may use the following model and notation for most answers:
2 period lives.
l commodities
1 person per generation
$u_0(x_0), x_0 = x_0^1, \omega_0 = \omega_0^1, m_0 = m_0^1$ for $t = 0$
$u_t(x_t), x_t = (x_t^t, x_t^{t+1}), \omega_t = (\omega_t^t, \omega_t^{t+1}), m_t = (m_t^t, m_t^{t+1})$ for t = 1,2,...

(a) Define Pareto Optimality (PO).

(b) Define Weak Pareto Optimality (WPO).

(c) Define Short Run Pareto Optimality (SRPO).

(d) Give the precise relationships among PO, WPO, and SRPO.

(e) State the First Welfare Theorem for OG.

(f) State the Second Welfare Theorem for OG in two ways:

   (a) allowing only for reassigning the ω's
   (b) allowing only for money taxes, $m_0, m_1, ..., m_t, ...$

(g) What is absence of money illusion? When does it obtain? How does it differ from the quantity theory?

(h) Precisely state the Phelps-Koopmans efficiency theorem.

(i) What are the connections between Phelps-Koopmans efficiency and welfare in the OG economy?

(j) Construct an economy in which oversaving is not the only source of inefficiency, namely in which short-run efficiency obtains but long-run efficiency does not. [Hint: You probably need more than one capital good.]

Solution:

3a. The allocation $x = (x_0, x_1, ..., x_t, ...)$ is said to be Pareto-Optimal (PO) if there is no alternative allocation $y = (y_0, y_1, ..., y_t, ...)$ with the property that

$$\sum_t y_t = \sum_t x_t$$

and

$$u_t(y_t) \geq u_t(x_t) \text{ with at least one strict inequality } \text{ for } t = 0, 1, ...$$

3b. The allocation $x = (x_0, x_1, ..., x_t, ...)$ is said to be weakly Pareto-optimal (WPO) if there is no $y = (y_0, y_1, ..., y_t, ...)$ with the property

$$\sum_t y_t = \sum_t x_t,$$

$$y_t = x_t \text{ except for a finite number of } t,$$

and

$$u_t(y_t) \geq u_t(x_t) \text{ with at least one strict inequality } \text{ for } t = 0, 1, ...$$

3c. The allocation $x = (x_0, x_1, ..., x_t, ...)$ is said to be short-run Pareto-optimal (SRPO) if there is no $y = (y_0, y_1, ..., y_t, ...)$ and $t' \geq 0$ with the property

$$\sum_t y_t = \sum_t x_t,$$

$$y_t = x_t \text{ for every } t \geq t',$$

and

$$u_t(y_t) \geq u_t(x_t) \text{ with at least one strict inequality } \text{ for } t = 0, 1, ...$$

3d. If x is PO, then x is also WPO. The allocation x is SRPO if and only if x is WPO.

3e. Every competitive equilibrium allocation is weakly Pareto-optimal.

3f. (a) Every weakly Pareto-optimal allocation can be achieved as a competitive allocation associated with some suitably assigned endowments.

(b) Every weakly Pareto-optimal allocation can be achieved as a competitive allocation associated with suitably assigned money taxes.

3g. Absence of money illusion is about sets. The quantity theory of money is about points.

For example, if money transfers m are consistent with the price p^m, then when money transfers are λm, by absence of money illusion there is a corresponding equilibrium with money price $\frac{p^m}{\lambda}$.

On the other hand, the quantity theory of money is stronger (and not empirically justified). It says that if m is multiplied by λ, then the equilibrium price *must* be $\frac{p^m}{\lambda}$.

3h. Consider the 1-sector model

$$\dot{k}(t) = f(k(t) - (n+\mu)k(t) - c(t))$$

with initial condition $k(0) = k_0$.

Definition: A program $\{c(t), k(t)\}_{t=0}^{\infty}$ is **efficient** if there is no other program $\{\hat{c}(t), \hat{k}(t)\}_{t=0}^{\infty}$ in which $\hat{k}(0) \leq k_0$, $\hat{c}(t) \geq c(t)$ with strict inequality on an interval $[t_1, t_2]$ where $0 \leq t_1 < t_2 < \infty$. Otherwise $\{c(t), k(t)\}$ is inefficient.
Define k_{gr} by $f(k) = n + \mu$ or $r = n$.

PK Theorem: Let ϵ be a positive scalar (independent of time). If $k(t) > k_{gr} + \epsilon$ for $t > t_0$, $0 \leq t_0 < \infty$, then $\{c(t), k(t)\}$ is inefficient.

3i. If $r(t) < n + \varepsilon$, then we have PK inefficiency. In the Samuelson OG model $n = 0$. If $r(t) < -\varepsilon$, then we have failure of PO.

3j.

$$\begin{aligned} Y &= C + Z_1 + Z_2 \\ Y &= F(K_1 + 2K_2, L) \\ \dot{K}_i &= Z_i \text{ for } i = 1, 2 \\ \dot{L} &= nL \end{aligned}$$

One should always set $Z_1 = 0$ for infinite planning, but if you are short-run planning, the exogenous terminal condition on K_1 might require stretches of $Z_1 > 0$.

4 Optimal Growth

$$Y = 10K^{1/3}L^{2/3}$$
$$Y = C + Z$$
$$\dot{K} = Z - K/10$$
$$\frac{\dot{L}}{L} = 0.02$$
$$\delta = 0.01$$
$$U\big(C/L\big) = \log\big(C/L\big)$$
$$K(0)/L(0) = 2$$
$$T = \infty$$

4a. Derive the Euler equation.

4b. Calculate k_{gr} and k_{mgr}.

4c. Draw the phase diagram.

4d. Do the full dynamic analysis.

Let $T < \infty$

4e. How is the problem altered?

4f. Precisely state the turnpike theorem.

Solution:

4a. Preliminaries: Define $C(t)/L(t) = c(t),\ K(t)/L(t) = k(t),$ and $Y(t)/(L(t) = y(t)$. Since $\delta > 0$, the integral is positive and the problem is bounded and well defined. Rearranging market clearing, we formulate the resource constraint in per capita terms:

$$\begin{aligned} Y(t) &= C(t) + Z(t) \\ &= C(t) + \dot{K}(t) + K(t)/10 \\ \Leftrightarrow\ C(t) &= 10K(t)^{1/3}L(t)^{2/3} - K(t)/10 - \dot{K}(t) \\ \Leftrightarrow\ c(t) &= 10k(t)^{1/3} - k(t)/10 - \dot{K}(t)/L(t) \end{aligned}$$

Using that $\frac{\dot{K}}{L} = \dot{k} + nk$,

$$\begin{aligned} c(t) &= 10k(t)^{1/3} - k(t)/10 - \dot{k} - nk \\ \Leftrightarrow\ c(t) &= 10k(t)^{1/3} - 0.12k(t) - \dot{k} \end{aligned}$$

Back to the notation in the lecture notes, note that $\lambda \equiv (n + \mu) = 0.02 + 0.1 = 0.12$. Our problem is as follows:

$$\begin{aligned} &\max_{c,k} \int_0^\infty \log\big(c(t)\big)e^{-\delta t}dt \quad s.t.\ \ c(t) = 10k(t)^{1/3} - 0.12k(t) - \dot{k(t)},\ \ k(0) = 2 \\ \Leftrightarrow &\max_{k,\dot{k}} \int_0^\infty \log\big(10k(t)^{1/3} - 0.12k(t) - \dot{k(t)}\big)e^{-\delta t}dt \quad s.t.\ \ k(0) = 2 \end{aligned}$$

From the lecture notes, we know that for any extremized function of the form

$$\int_{t_0}^{t_1} \phi\big(x(t), \dot{x}(t), t\big)dt, \quad s.t.\ \ x(t_0) = x_0,\ \ x(t_1) = x_1$$

the Euler Equation will take the functional form $\phi_{x(t)} = \frac{d}{dt}\phi_{\dot{x}(t)}$.

Here $\phi(\cdot) = \log(\cdot)$, state variable $x(t) = k(t)$, and $\dot{x}(t) = \dot{k}(t)$.

Trick: Define $U_{c(t)} = -\frac{1}{10k(t)^{1/3} - 0.12k(t) - \dot{k(t)}} = q(t)$.

As preliminaries to characterizing the Euler Equation in this form, we have:

$$\phi_{k(t)} = \Big(q(t)\big(f_k - 0.12\big)\Big)e^{-\delta t} \tag{1}$$

$$\begin{aligned} \phi_{\dot{k}(t)} &= -q(t)e^{-\delta t} \\ \frac{d}{dt}\phi_{\dot{k}(t)} &= -\dot{q}e^{-\delta t} + \delta e^{-\delta t} \end{aligned}$$

$$= (-\dot{q} + \delta q)e^{-\delta t} \tag{2}$$

To characterize the Euler Equation by equating (1) and (2)

$$
\begin{aligned}
q(t)\big(f_k - 0.12\big) &= (-\dot{q} + \delta q) \\
\Leftrightarrow \frac{\dot{q(t)}}{q(t)} &= \delta + \lambda - f_k(t) \\
\Leftrightarrow \frac{\dot{q(t)}}{q(t)} &= .13 - \frac{10}{3}k(t)^{-2/3}
\end{aligned}
$$

The dynamics of the economy are fully characterized by initial condition $k_0 = 2$, the Euler Equation, and the TVC: $\lim_{T\to\infty} qe^{-\delta T}k(T) = 0$.

4b. From the lecture notes, we know that $f'(k_{gr}) = \lambda = \mu + n$ and $f'(k_{mgr}) = \delta + \lambda = \delta + \mu + n$. Thus

$$
\begin{aligned}
\frac{10}{3}k_{gr}^{-2/3} &= 0.12 \\
\Leftrightarrow k_{gr} &= \Big(\frac{3}{10}(0.12)\Big)^{-3/2} \\
&\approx 146.4 \\
\frac{10}{3}k_{mgr}^{-2/3} &= 0.13 \\
\Leftrightarrow k_{mgr} &= \Big(\frac{3}{10}(0.13)\Big)^{-3/2} \\
&\approx 129.8
\end{aligned}
$$

4c. See notes from TA Section on Friday, December 4, 2015. The end result should resemble Figure 1 from Cass (ECTA, 1966)

DAVID CASS

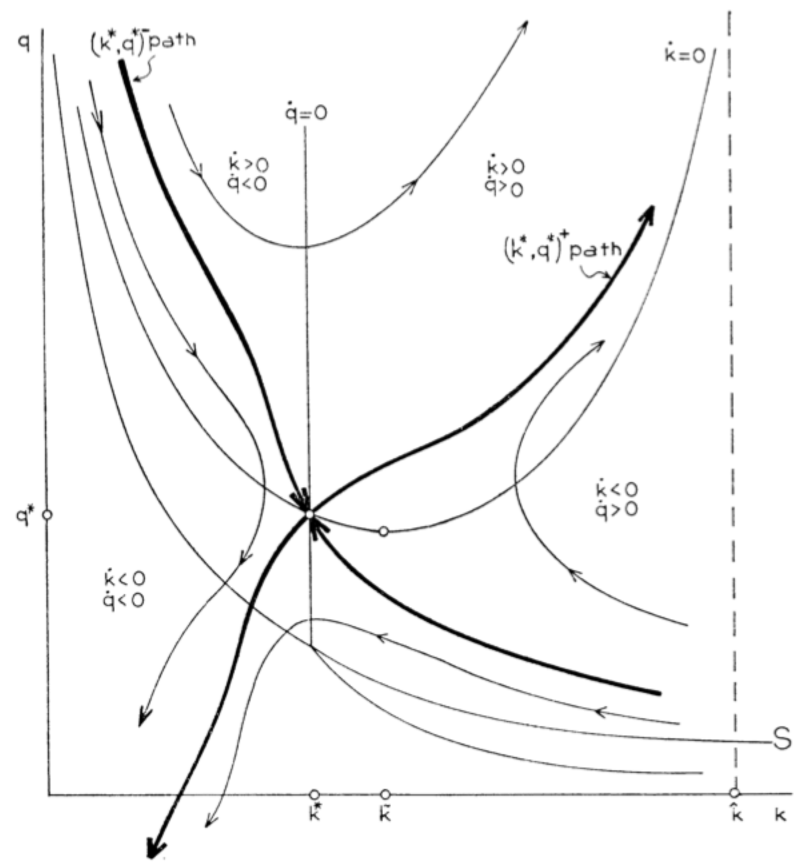


FIGURE 1.—The nature of the solutions to the optimality conditions.

4d. Being in the infinite horizon, the economy moves along the stable manifold, characterized by the Euler Equation, from $k_0 = k(0) = 2$ to $k_{mgr} = 129.8$, all in the second quadrant.

4e. In a finite horizon problem, the introduction of a terminal condition k_T would almost certainly require optimal policy to eventually deviate from k_{mgr}, so that the path resembles a turnpike: A transition from k_0 to k_{mgr} along the stable manifold, maintenance of k within a small neighborhood of k_{mgr}, then a late transition from the ε ball around k_{mgr} to k_T. And as a general matter, in the finite horizon, it is no longer necessary for δ to be positive.

4f. Optimum growth turnpike theorem à la Cass (1966):

Given any positive $\varepsilon > 0$, define the closed ε-neighborhood $N(\varepsilon)$ of the modified golden rule growth path by

$$N(\varepsilon) = \big\{(k, q) : |k - k_{mgr}| \leq \varepsilon,\ |q - q^*| \leq \varepsilon\big\}.$$

Then for the unique optimum growth path $\big\{(k(t), q(t)) : 0 \leq t \leq T\big\}$ specified by the initial and terminal parameters (k_0, k_T, T), there exist two finite times $0 \leq T_1 < \infty$ and $0 \leq T_2 < \infty$, $T_1 = T_1(\varepsilon, k_0)$ and $T_2 = T_2(\varepsilon, k_T)$, such that $(k(t), q(t)) \in N(\varepsilon)$ whenever $T_1 \leq t \leq T - T_2$.

Translation: Thus, for a "sufficiently long period" $T > T_1 + T_2$, the theorem asserts a strong turnpike property for optimum growth over any planning period [0, T], in the sense that it states that such growth occurs within an arbitrarily small neighborhood of the "best" balanced growth path, except possibly over some initial or terminal phase.

5 Overlapping Generations, III

$$\omega_0^1 = B > 0 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (A, B) > 0 \text{ for } t = 1, 2, ...$$
$$u_0(x_0^1) = (x_0^1)^D \text{ where } D > 0, \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = (x_t^t)^C (x_t^{t+1})^D \text{ where } (C, D) > 0, \text{ for } t = 1, 2, ...$$

For what values of $(A, B, C, D) > 0$ is the economy (a) Sumuelsonian or (b) Ricardian?

Solution:

The utility maximization problems of the initial old generation and the date-t generations are preserved under a log transformations of their respective utility functions, i.e.,

$$u_0(x_0^1) = D \log(x_0^1), \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = C \log(x_t^t) + D \log(x_t^{t+1}), \text{ for } t = 1, 2, ...$$

We have previously shown that we have Ricardo whenever $CB \geq DA$, and we get Samuelson whenever $CB < DA$ for generic OLG economies with stationary endowments

$$\omega_0^1 = B > 0 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (A, B) > 0 \text{ for } t = 1, 2, ...$$

and stationary log preferences

$$u_0(x_0^1) = D \log(x_0^1) \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = C \log(x_t^t) + D \log(x_t^{t+1}) \text{ for } t = 1, 2, ...$$

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