Economics 4905
Financial Fragility and the Macroeconomy
Cornell University
Fall 2017
Solutions to the Practice Questions for Prelim 2

1. Money Taxation with Two Monies

Consider an economy with a single commodity, $\ell = 1$, chocolate. There are 5 consumers, so $n = 5$. The endowments are defined as

$$\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$$
$$= (30, 15, 24, 40, 10)$$

measured in ounces of chocolate.

Consider a scenario where there are 2 monies, red dollars R and blue dollars B, with respective chocolate prices of money, $P^R \geq 0$ and $P^B \geq 0$.

In each of the following cases, solve for the equilibrium exchange rate between B and R. Do these depend on the endowments ω? Give the economic explanation for your answer.

For each of the 3 cases, solve for the set of equilibrium allocations.

a) $\tau^R = (1, 2, 3, 0, -2)$ and $\tau^B = (1, 0, -1, 0, -2)$

Solution:

Recall that the exchange rate is:

$$\frac{P^R}{P^B} = -\frac{\sum_h \tau_h^B}{\sum_h \tau_h^R}$$

Summing up the taxes for both monies:

$$\sum_h \tau_h^R = 1 + 2 + 3 + 0 + (-2) = 4$$
$$\sum_h \tau_h^B = 1 + 0 + (-1) + 0 + (-2) = -2$$

The exchange rate is then

$$\frac{P^R}{P^B} = -\frac{\sum_h \tau_h^B}{\sum_h \tau_h^R} = -\frac{-2}{4} = \frac{1}{2}$$

For a given equilibrium price of money P^R, the equilibrium allocations are:

$$x = (30, 15, 24, 40, 10) - P^R(1, 2, 3, 0, -2) - \frac{1}{2}P^B(1, 0, -1, 0, -2)$$
$$= (30 - 1.5P^R, 15 - 2P^R, 24 - 2.5P^R, 40, 10 + 3P^R)$$

Since $\bar{P}^R = 7.5$, so

$$\mathcal{P}^R = [0, 7.5)$$

The set of equilibrium allocations are

$$\{(30 - 1.5P^R, 15 - 2P^R, 24 - 2.5P^R, 40, 10 + 3P^R) | P^R \in [0, 7.5)\}$$

b) $\tau^R = (1, 1, 0, -1, -2)$ and $\tau^B = (1, 1, 1, 0, -5)$

Solution:
Summing up the taxes for both monies:

$$\sum_h \tau_h^R = 1 + 1 + 0 + (-1) + (-2) = -1$$
$$\sum_h \tau_h^B = 1 + 1 + 1 + 0 + (-5) = -2$$

Since we need

$$P^R \sum_h \tau_h^R + P^B \sum_h \tau_h^B = 0$$

the only way this equation could hold is that

$$P^R = P^B = 0$$

which makes the exchange rate $\frac{P^R}{P^B}$ indeterminate. Since prices of both monies are 0, the equilibrium allocation is just equal to the endowment:

$$x = (30, 15, 24, 40, 10)$$

c) $\tau^R = (1, 1, 2, 0, -4)$ and $\tau^B = (5, 0, -1, -2, -2)$

Solution:
Summing up the taxes for both monies:

$$\sum_h \tau_h^R = 1 + 1 + 2 + 0 + (-4) = 0$$
$$\sum_h \tau_h^B = 5 + 0 + (-1) + (-2) + (-2) = 0$$

Again, the exchange rate is indeterminate because both taxes are balanced. Given equilibrium prices of money P^R and P^B, the equilibrium allocations are

$$\begin{aligned} x &= (30, 15, 24, 40, 10) - P^R(1, 1, 2, 0, -4) - P^B(5, 0, -1, -2, -2) \\ &= (30 - P^R - 5P^B, 15 - P^R, 24 - 2P^R + P^B, 40 + 2P^B, 10 + 4P^R + 2P^B) \end{aligned}$$

The equilibrium prices has to satisfy all of the following conditions:

$$\begin{aligned} 30 - P^R - 5P^B &\geq 0 \\ 15 - P^R &\geq 0 \\ 24 - 2P^R + P^B &\geq 0 \end{aligned}$$

Graphing the three inequalities above should give us that only the following two are necessary:

$$30 - P^R - 5P^B \geq 0$$
$$24 - 2P^R + P^B \geq 0$$

The set of equilibrium allocations is

$$\{(30 - P^R - 5P^B, 15 - P^R, 24 - 2P^R + P^B, 40 + 2P^B, 10 + 4P^R + 2P^B)|$$
$$30 - P^R - 5P^B \geq 0, 24 - 2P^R + P^B \geq 0, P^R \geq 0, P^B \geq 0\}$$

2. Inter-temporal Economics

2.1 Futures Market Economy

Consider a 1 good, 2 individuals ($h = 1, 2$), 2 periods ($t = 1, 2$) futures market economy.

$$u_h = (x_h^1)^\alpha (x_h^2)^\beta \text{ for } h = 1, 2$$
$$\text{Mr 1:} \quad \omega_1 = (100, 50) = (\omega_{11}, \omega_{12})$$
$$\text{Mr 2:} \quad \omega_2 = (50, 100) = (\omega_{21}, \omega_{22})$$

Set up the CP and CE for when there are (only) perfect futures markets.

Solve for the CE allocations, the CE prices, the interest factors R, and the CE interest rate r for the following cases:

(a) $\alpha = 1, \beta = 1$

(b) $\alpha = 1, \beta = 5$

(c) $\alpha = 5, \beta = 1$

Discuss the economics of your answers to parts (a), (b) and (c).

Solution:

Consumer Problem (CP):

$$\max_{x_h^1, x_h^2} \quad (x_h^1)^\alpha (x_h^2)^\beta$$
$$\text{subject to} \quad p^1 x_h^1 + p^2 x_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2$$

A competitive equilibrium (CE) is a set of prices $(p^1, p^2) \geq 0$ such that CP is satisfied for all consumers $h = 1, 2$ and goods market clear in both periods.

Using Lagrangian to solve the CP, the first order conditions can be summarized as

$$\beta = \frac{p^2}{p^1} \frac{x_h^2}{x_h^1}$$

Plugging this into the budget constraint in the CP gives

$$x_h^1 = \frac{\alpha}{p^1(\alpha+\beta)}(p^1\omega_h^1 + p^2\omega_h^2)$$
$$x_h^2 = \frac{\beta}{p^2(\alpha+\beta)}(p^1\omega_h^1 + p^2\omega_h^2)$$

When $\alpha = 1, \beta = 1$,

$$x_1^1 = \frac{\alpha}{p^1(\alpha+\beta)}(p^1\omega_1^1 + p^2\omega_1^2) = \frac{1}{p^1(1+1)}(100p^1 + 50p^2) = 50 + 25\frac{p^2}{p^1}$$
$$x_1^2 = \frac{\beta}{p^2(\alpha+\beta)}(p^1\omega_1^1 + p^2\omega_1^2) = \frac{1}{p^2(1+1)}(100p^1 + 50p^2) = 50\frac{p^1}{p^2} + 25$$
$$x_2^1 = \frac{\alpha}{p^1(\alpha+\beta)}(p^1\omega_2^1 + p^2\omega_2^2) = \frac{1}{p^1(1+1)}(50p^1 + 100p^2) = 25 + 50\frac{p^2}{p^1}$$
$$x_2^2 = \frac{\beta}{p^2(\alpha+\beta)}(p^1\omega_2^1 + p^2\omega_2^2) = \frac{1}{p^2(1+1)}(50p^1 + 100p^2) = 25\frac{p^1}{p^2} + 50$$

Set $p^1 = 1$, from market clearing condition for good 1:

$$\sum_h x_h^1 = \sum_h \omega_h^1$$

$$50 + 25p^2 + 25 + 50p^2 = 100 + 50$$
$$p^2 = 1$$

Since $p^1 = p^2 = 1$, we have $R = 1$ and $r = 0$. The CE allocations are

$$x_h^1 = x_h^2 = 75 \quad \text{for } h = 1, 2$$

When $\alpha = 1, \beta = 5$,

$$(x_1^1, x_1^2) = (\frac{50}{3} + \frac{25}{3}\frac{p^2}{p^1}, \frac{250}{3}\frac{p^1}{p^2} + \frac{125}{3})$$
$$(x_1^1, x_1^2) = (\frac{25}{3} + \frac{50}{3}\frac{p^2}{p^1}, \frac{125}{3}\frac{p^1}{p^2} + \frac{250}{3})$$

Set $p^1 = 1$, from market clearing condition for good 1:

$$\frac{50}{3} + \frac{25}{3}p^2 + \frac{25}{3} + \frac{50}{3}p^2 = 100 + 50$$
$$p^2 = 5$$

We have $R = 1/5$ and $r = -4/5$. The CE allocations are

$$(x_1^1, x_1^2) = (\frac{175}{3}, \frac{175}{3})$$

$$(x_2^1, x_2^2) = (\frac{275}{3}, \frac{275}{3})$$

When $\alpha = 5, \beta = 1$:

$$(x_1^1, x_1^2) = (\frac{250}{3} + \frac{125}{3}\frac{p^2}{p^1}, \frac{50}{3}\frac{p^1}{p^2} + \frac{25}{3})$$

$$(x_1^1, x_1^2) = (\frac{125}{3} + \frac{250}{3}\frac{p^2}{p^1}, \frac{25}{3}\frac{p^1}{p^2} + \frac{50}{3})$$

Set $p^1 = 1$, from market clearing condition for good 1:

$$\frac{250}{3} + \frac{125}{3}p^2 + \frac{125}{3} + \frac{250}{3}p^2 = 100 + 50$$

$$p^2 = \frac{1}{5}$$

We have $R = 5$ and $r = 4$. The CE allocations are

$$(x_1^1, x_1^2) = (\frac{275}{3}, \frac{275}{3})$$

$$(x_2^1, x_2^2) = (\frac{175}{3}, \frac{175}{3})$$

Notice that in all three cases, we have $R = \alpha/\beta$. There is also perfect consumption smoothing because the aggregate endowments are constant across time. As $\beta > \alpha$, demand for period 2 consumption is higher. Consumption in period 2 gets more expensive. The consumer with higher endowment in period 2 ends up consuming more in both periods in equilibrium.

2.2 Money Market Economy

Consider a 1 good, 2 individuals, 1 inside money money market economy. The utility function and endowments of the consumers are identical as in Part (2.1).

Set up the CP and the CE in this problem. Show that the CE allocations in (2.1) are also a CE allocations in (2.2). Identify in (a), (b) and (c) which individual is a borrower and which one is a lender. Discuss the economics of your answers.

Show that there is a CE allocation in Example B that is not a CE allocation in Example A. Discuss the economics.

Solution:

Consumer Problem (CP):

$$\begin{aligned} \max_{x_h^1, x_h^2} \quad & (x_h^1)^\alpha (x_h^2)^\beta \\ \text{subject to} \quad & p^1 x_h^1 + p^{m^1} m_h^1 = p^1 \omega_h^1 \\ & p^2 x_h^2 + p^{m^2} m_h^2 = p^2 \omega_h^2 \end{aligned}$$

A competitive equilibrium (CE) is a set of prices $(p^1, p^2, p^{m^1}, p^{m^2}) \geq 0$ such that CP is satisfied for all consumers $h = 1, 2$ and goods market clear in both periods.

We know from homework that $p^{m^1} = p^{m^2}$ and $m_h^1 + m_h^2 = 0$. The two constraints in the CP can be combined into

$$p^1 x_h^1 + p^m m_h^1 + p^2 x_h^2 + p^m(-m_h^1) = p^1 \omega_h^1 + p^2 \omega_h^2$$
$$p^1 x_h^1 + p^2 x_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2$$

which is exactly the constraint in the futures market economy. Therefore, the two problems are equivalent. To find out whether a consumer is a borrower or lender, we simply compute $\omega_h^1 - x_h^1$. If this number is negative, the consumer is a borrower. If this number is positive, the consumer is a lender.

Following these steps, we should get that Mr 1 is always a lender and Mr 2 is always a borrower. This is because Mr 1 has high endowment in period 1 and low endowment in period 2. Therefore he tends to lend in period 1 to trade some consumption in period 1 for consumption in period 2.

The CE that exists in (2.2) but not in (2.1) is the special case when $p^m = 0$. In this case, the constraints in (2.2) become

$$p^1 x_h^1 = p^1 \omega_h^1$$
$$p^2 x_h^2 = p^2 \omega_h^2$$

Good prices p^1 and p^2 can be any positve number. The only choice the consumer has is to consume his endowment in each period. This constitutes a CE in the money market economy but not the futures market economy.

2.3 Comparing the Two Economies

What is the difference between the futures market and the money market? Which model is more realistic?

Solution:
Every futures market equilibrium allocation can be decentralized as a money market equilibrium. The money market is more realistic: we usually borrow and lend in terms of dollars. The money market is more fragile: worthless money is always possible.

3. Bank Runs

Consider the Diamond-Dybvig bank run model. The probability λ of being impatient is 50%. The utility function is:

$$u(c) = 10 - \frac{1}{(0.5)\sqrt{c}}.$$

The rate of return to the asset harvested late is 400%, i.e.,

$$R = 5.$$

(a) What is the depositor's *ex-ante* expected utility W as a function of c_1, consumption in period 1, and c_2, consumption in period 2?

Solution:
$W = \lambda u(c_1) + (1-\lambda)u(c_2)$
$W = \lambda(10 - \frac{1}{0.5\sqrt{c_1}}) + (1-\lambda)(10 - \frac{1}{0.5\sqrt{c_1}}) = \frac{1}{2}\left(10 - \frac{1}{0.5\sqrt{c_1}}\right) + \left(1 - \frac{1}{2}\right)\left(10 - \frac{1}{0.5\sqrt{c_2}}\right)$ So

$$W = 5 - \frac{1}{\sqrt{c_1}} + 5 - \frac{1}{\sqrt{c_2}} = 10 - \frac{1}{\sqrt{c_1}} - \frac{1}{\sqrt{c_2}}$$

(b) Show that the depositor prefers consumption smoothing.

Solution:

$u''(c) < 0$. Hence, $u(c)$ is a strictly concave. A concave functions lies above its chords (Jensen's inequality):
$u(\lambda c_1 + (1-\lambda)c_2) > \lambda u(c_1) + (1-\lambda)u(c_2)$ when $c_1 \neq c_2$,

That is she prefers $(\bar{c}, \bar{c})$ to (c_1, c_2) where $\bar{c} = \lambda c_1 + (1-\lambda)c_2$.

(c) Why can't she insure on the market against liquidity shocks?

Solution:
Her type is purely her own private information. The insurance company would not trust her to report her type truthfully. She would say that she is impatient even if she is not.

Assume that her endowment is 100 and that she deposits her entire endowment in the bank.

(d) What is her utility W in autarky?
Solution:

$W_{autarky} = \lambda u(100) + (1-\lambda)u(500) = 9.855$

(e) What is her utility W under perfect smoothing, i.e. when $c_1 = c_2$?
Solution:

$W_{perfect-smoothing} = u(\lambda \cdot 100 + (1-\lambda) \cdot 500) = 9.88$

(f) What is the bank's resource constraint RC? Write this down precisely. Explain this in words.
Solution:

$$(0.50)(d_2) \leq (\omega - (0.50)d_1)R$$

Or

$$(0.50)(d_2) \leq (100 - (0.50)d_1) \cdot 5$$

Period-2 withdrawals cannot exceed period-2 bank resources. This may be re-written as

$$d_2 \le 2(100 - 0.50 \cdot d_1) \cdot 5$$

$$d_2 \le 1000 - 5 \cdot d_1$$

(g) What is the incentive problem? Write this down precisely and explain in words the incentive constraint IC.
Solution:

$d_1 \le d_2$ (ICC)

If ICC does not hold, everyone will seek to withdraw in period 1.

(h) Find the optimal deposit contract for this bank. What is W if there is no run?
Solution:

$$\arg\max_{d_1,d_2} \lambda u(d_1) + (1-\lambda)u(d_2)$$

subject to RC and ICC.
Using Lagrangian Optimization (see the lecture notes, or Problem 2), we may find that

$$\frac{u'(d_1^*)}{u'(d_2^*)} = R$$

Since $u(c) = 10 - \frac{1}{(0.5)\sqrt{c}}$, it follows that $u'(c) = \frac{d}{dc}\left(10 - 2c^{-1/2}\right) = -2 \cdot \left(-\frac{1}{2}\right) c^{-3/2} = c^{-3/2}$,

$$\frac{(d_1^*)^{-3/2}}{(d_2^*)^{-3/2}} = \left(\frac{d_1^*}{d_2^*}\right)^{-3/2} = R \quad \Rightarrow \quad \left(\frac{d_2^*}{d_1^*}\right)^{3/2} = R$$

And therefore

$$\frac{d_2^*}{d_1^*} = R^{2/3} \quad \Rightarrow \quad d_2^* = d_1^* R^{2/3}$$

Recalling the resource constraint, $d_2^* = 1000 - 5 \cdot d_1^*$. Thus,

$$1000 - 5 \cdot d_1^* = d_1^* \cdot 5^{2/3} \quad \Rightarrow d_1^*(5 + 5^{2/3}) = 1000$$

So

$$d_1^* = \frac{1000}{(5 + 5^{2/3})} = 126.2$$

$$d_2^* = 1000 - 5 \cdot 126.2 = 369.0$$

As such, $d_1^* = 126.2, d_2^* = 369.0$
$W_{no-run} = 0.5u(d_1^*) + 0.5u(d_2^*) = 9.859$

(i) Why is there a run equilibrium for this bank?
Solution:
$d_1^* = 126.20 > 100$

If every depositor attempted to withdraw at once (not just the impatient ones, but the impatient ones, too), then the bank will not be able to pay everyone at once

(j) Calculate the following numerical values of *ex-ante* utility W and and rank them in numerical ascending order: $W_{autarky}$, $W_{perfect\ smoothing}$, $W_{no\ run}$, W_{run}.
Solution:
$W_{run} = (100/d_1^*)u(d_1^*) = 7.783$
$W_{run} < W_{autarky} < W_{no-run} < W_{perfectsmoothing}$

(k) Assume that the run probability s is 2%. Will individuals deposit in this bank? That is, will they accept this banking contract? Explain.
Solution:
$(.02)W_{run} + (.98)W_{no-run} = (0.02)(7.783) + (0.98)(9.86) = 9.812 < 9.855 = W_{autarky}$
Consumers will *not* deposit at the bank if there is a 2% run probability.

4. The Overlapping Generations Model

The model is set up as follow:

- 2 period lives
- 1 commodity per period, $\ell = 1$
- Stationary environment
- 1 person per generation

The utility functions are given as:

$$u_0(x_0^1) = x_0^1$$
$$u_t(x_t^t, x_t^{t+1}) = (x_t^t)^\alpha (x_t^{t+1})^\beta \text{ for } t = 1, 2, \ldots$$

The endowments are 1 unit each person each period she is alive:

$$\omega_0^1 = \omega_t^t = \omega_t^{t+1} = 1 \text{ for } t = 1, 2, \ldots$$

Define:

$$z^t = x_t^t - \omega_t^t$$
$$z^{t+1} = \omega_t^{t+1} - x_t^{t+1}$$

Case 1: $\alpha = 1, \beta = 10, m_0^1 = 1, m_s^t = 0$ otherwise
Case 2: $\alpha = 1, \beta = 0.2, m_0^1 = 1, m_s^t = 0$ otherwise

For each of the above cases, solve for the following:

a) The equilibrium demand (x_t^t, x_t^{t+1})

Solution:
The consumer problem is:

$$\max_{x_t^t, x_t^{t+1}} \quad (x_t^t)^\alpha (x_t^{t+1})^\beta$$
$$\text{subject to} \quad p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1}$$

The solution to the consumer problem is

$$x_t^t = \frac{\alpha}{\alpha+\beta} \frac{p^t \omega_t^t + p^{t+1}\omega_t^{t+1}}{p^t}$$
$$x_t^{t+1} = \frac{\beta}{\alpha+\beta} \frac{p^t \omega_t^t + p^{t+1}\omega_t^{t+1}}{p^{t+1}}$$

Case 1:

$$x_t^t = \frac{1}{11}\left(1 + \frac{p^{t+1}}{p^t}\right)$$
$$x_t^{t+1} = \frac{10}{11}\left(\frac{p^t}{p^{t+1}} + 1\right)$$

Case 2:

$$x_t^t = \frac{5}{6}\left(1 + \frac{p^{t+1}}{p^t}\right)$$
$$x_t^{t+1} = \frac{1}{6}\left(\frac{p^t}{p^{t+1}} + 1\right)$$

b) The offer curve (OC)

Solution:

From Problem Set 4, the offer curve is:

$$z^{t+1} = \frac{\alpha \omega_t^{t+1} z^t}{\beta \omega_t^t - (\alpha+\beta) z^t}$$

Case 1:

$$z^{t+1} = \frac{z^t}{10 - 11 z^t}$$

Case 2:

$$z^{t+1} = \frac{5 z^t}{1 - 6 z^t}$$

c) The steady states

Solution: Refer to PS4 solutions

d) The set of equilibrium money prices, $\mathcal{P}^m$

Solution: Refer to PS4 solutions

e) The full dynamic analysis, including the stability of steady states

Solution: Refer to PS4 solutions

5. Quantity Theory of Money

What is the difference between "the quantity theory of money" and "the absence of money illusion"? Be precise. What is the importance of this for financial fragility and economic policy? Be brief.

Solution:
Under the quantity theory of money, if outside money is doubled, the price of money will halve (and the price level for real goods, by extension, will double). In contrast, with an absence of money illusion, it is only a possibility that the same fiscal policy change will halve the price of money and double the price level. The general price level is not tied down if only the absence of money illusion holds, allowing for fragility of price level responses to monetary policy and other policies.