Economics 4905: Lecture 9

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Substitution Method

Consider the following problem:

$$\max_{x_h^1, x_h^2} \quad u_h(x_h^1, x_h^2)$$

$$\text{s.t.} \quad p^1 x_h^1 + p^2 x_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2 \equiv \omega_h$$

Substitution Method

Example:

$$\begin{aligned} \max_{x_h^1, x_h^2} \quad & u_h(x_h^1, x_h^2) = \log x_h^1 + \beta \log x_h^2 \\ \text{s.t.} \quad & p^1 x_h^1 + p^2 x_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2 \equiv \omega_h \end{aligned}$$

Solving the constraint for x_h^2 in terms of x_h^1:

$$p^2 x_h^2 = \omega_h - p^1 x_h^1$$

$$x_h^2 = \frac{\omega - p^1 x_h^1}{p^2}$$

Substitution Method

Plugging into the utility function:

$$u_h = \log x_h^1 + \beta \log \left(\frac{\omega - p^1 x_h^1}{p^2} \right)$$

Taking the derivative with respect to x_h^1:

$$u_h' = \frac{1}{x_h^1} - \frac{\beta p^1}{p^2 x_h^2} = 0$$

Combining this with the budget constraint gives:

$$p^1 x_h^1 = \frac{p^2 x_h^2}{\beta}$$
$$p^1 x_h^1 = \frac{\omega}{1+\beta}$$
$$p^2 x_h^2 = \frac{\omega\beta}{1+\beta}$$

Rule of Lagrange

Consider the following problem:

$$\max_{x,y} \quad f(x,y)$$
$$\text{s.t.} \quad \phi(x,y) = 0 \rightarrow y = g(x)$$

The Lagrangian is:

$$\Lambda = f(x,y) + \lambda\phi(x,y)$$

The first-order conditions:

$$\Lambda_x = f_x + \lambda\phi_x = 0$$
$$\Lambda_y = f_y + \lambda\phi_y = 0$$
$$\Lambda_\lambda = \phi(x,y) = 0$$

Proof of Lagrange

Rewriting the problem:

$$\max_{x} \quad f(x, g(x))$$

The first-order condition is

$$\begin{aligned} \frac{df}{dx} = f_x + f_y g'(x) &= 0 \\ f_x + f_y \left(\frac{dy}{dx}\right)_{\phi=0} &= 0 \\ f_x + f_y \left(\frac{\phi_x}{\phi_y}\right)_{\phi=0} &= 0 \end{aligned}$$

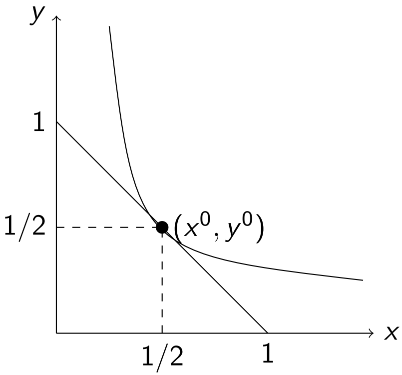
Proof of Lagrange

Let $\lambda = -\frac{f_y}{\phi_y}$. Then

$$\begin{aligned} f_x + \lambda\phi_x &= 0 \\ f_y + \lambda\phi_y &= 0 \\ \phi(x, y) &= 0 \end{aligned}$$

Example

$$\max_{x,y} \quad xy$$
$$\text{s.t.} \quad 1 - x - y = 0$$



Example

The Lagrangian for the problem is:

$$\Lambda = xy + \lambda(1 - x - y)$$

The first-order conditions are:

$$\begin{aligned} \Lambda_x &= y - \lambda = 0 \\ \Lambda_y &= x - \lambda = 0 \\ \Lambda_\lambda &= 1 - x - y \end{aligned}$$

Notice that $(x^0, y^0, \lambda^0) = (1/2, 1/2, 1/2)$ satisfies the first-order conditions.

Example

Now consider the alternative problem:

$$\max_{x,y} \quad xy$$
$$\text{s.t.} \quad (1 - x - y)^3 = 0$$

The diagram is exactly the same as the original problem. However, the Lagrangian is

$$\Lambda = xy + \lambda(1 - x - y)^3$$

and the first-order conditions are

$$\Lambda_x = y - 3\lambda(1 - x - y)^2$$
$$\Lambda_y = x - 3\lambda(1 - x - y)^2$$
$$\Lambda_\lambda = (1 - x - y)^3 = 0$$

However, there does not exist $\lambda^0 \in \mathbb{R}$ such that $(1/2, 1/2, \lambda^0)$ satisfies the first-order conditions. This is when Lagrangian fails.