The Diamond-Dybvig Revolution: Extensions Based on the Original DD Environment*

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Abstract

Phil Dybvig and Doug Diamond launched an unfolding revolution in the theory of banks and financial intermediaries. The DD model is general equilibrium, but not Walrasian. Rational beliefs about the beliefs of others are central. So extensions to DD can be expected to further open doors in monetary and macro–economics. In our contribution to the DD36 celebration, we extend the original DD banking model to cases in which depositor beliefs about the run probability depend in a rather general way on the banking contract. The equilibrium banking contract is a "fixed-point" where depositor beliefs drive the demand and the bank determines the supply. We rely heavily on game-theory style thinking.

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1 Introduction

We begin with the post-deposit game from the basic DD paper: The only feasible banking contract is the simple deposit contract; partial suspension of convertibility is ruled out. There is no aggregate uncertainty. There is a continuum of depositors. In an important break from the classic DD, we do not allow for deposit insurance.

Our post-deposit analysis is more global than that in DD. The so-called “optimal contract” (now known as the “unrestricted efficient contract”) is based on the assumption that runs cannot occur.[1] (It would be the optimal contract if the bank teller could detect when a depositor is falsely claiming to be impatient.)

The set of feasible contracts is divided into 3 subsets. For each of the most conservative contracts, there is a unique allocation. It is non-run. For each of the least conservative contracts, there is only one allocation. It is a run. For the other contracts, there are 2 allocations. One of the two is non-run; the other is run. For the most conservative contracts, the allocation is unique and run free. These contracts are dominant incentive compatible. For the middle contracts, there is one run allocation and one non-run allocation. These contracts are Bayesian incentive compatible, but not dominant incentive compatible. For the least conservative contracts, the sole allocation is a run. These contracts are not even Bayesian incentive compatible. The middle contracts are the pay dirt in the DD paper.

DD assume that the government provides deposit insurance to avoid bank runs. DD solve for the best run-proof contact. The post-deposit game is sufficient for analyzing this problem. A large and excellent literature follows DD in solving for best run-proof mechanisms. See, for example, Wallace [14] and Green and Lin [9]. This approach leads one to "narrow banking". It is also consistent with the position of Kotlikoff [10]. We, however, follow Peck and Shell [11] in solving for the best contract (or mechanism) which might or might not be run-proof. We believe this to be the natural way to look at the problem. It is rational to weigh risks versus gains. One crosses the street

[1] See Ennis and Keister [7].

even though there is a risk in doing so. Engineers do not build the safest bridge possible. To weigh the risk of runs, we turn to the pre-deposit game, in which the bank offers a contract and the consumer decides whether or not to deposit. We solve for the best contract, which might or might not tolerate runs. The other approach is to design the post-deposit game to be run proof and then to solve for the best of these contracts.

We assume that depositors assign a run probability s which is a function of the contract c. We assume (for now) that $s(c)$ is non-decreasing in c.[2] We assume for simplicity that depositors share the same beliefs. (We plan to introduce asymmetric depositor information in follow-up work.) The bank chooses the contract $c(s)$ as a function of the run probability s. The equilibrium outcome (c, s) is determined by the interaction of depositor beliefs and behavior, and the bank's contract design.

This equilibrium exits. If in addition $s(c)$ is continuous and smooth, the bank's problem is strictly concave so we know that the equilibrium is unique. We show by examples that if $s(c)$ is discontinuous or continuous-but-not smooth, then multiple equilibria are possible. When $s(c)$ is not constant in c, the bank's problem is altered by the feedback effect of the bank contract c on the depositor run risk s.

Several of the important results in this paper appear in the literature, but they are based on different elaborations of the banking environment or based on a non-banking model of coordination failure. We attempt to recognize the important literatures on which we rely in our footnotes and especially in our Summary. Please alert us to our errors of omission and commission.

2 The Model

2.1 Agents and Preferences

There are three periods ($t = 0$, 1 and 2) and a single good. There is a continuum of consumers with measure 1. In period 0, consumers are identical. In period 1, a liquidity shock turns a constant fraction λ of them into impatient

[2] The scalar c denotes "cash withdrawal in period 1".

consumers and the remaining $(1-\lambda)$ into patient consumers. An impatient consumer values only the period-1 good and a patient consumer values only the period-2 good. If a consumer consumes x units of the goods he values, his utility is given by the constant-relative-risk-aversion utility function $u(x)$[3], where

$$u(x) = \frac{(x+1)^{1-\gamma} - 1}{1-\gamma}, \gamma \geq 0.$$

The proportion of the impatient consumers λ is public information at time 0. In period 1, an individual consumer privately learns whether he is impatient or patient.

2.2 Endowments and Investment Technologies

Each consumer is endowed with one unit of the good in period 0. Two constant-returns-to-scale investment opportunities are available to the individuals and the banks. One is costless storage. The other is a long-term asset: a unit invested in period 0 generates one unit of the good if harvested in period 1, but $R > 1$ units of the good if harvested in period 2.

2.3 Banks and the Deposit Contract

We assume that there is free-entry banking. As in DD, we assume that banks offer only a simple deposit contract. In period 0, banks announce their deposit contracts and consumers choose whether to deposit or not. The deposit contract specifies c, the amount that a depositor can withdraw in period 1 up until the bank's resources have been depleted. Depositors who did not choose early withdrawal share evenly the resources remaining in the bank in the last period. Using backward induction, we start the analysis with the post-deposit game in period 1.

[3] $u(x)$ is not the standard CRRA utility function. We want to have a finite value of utility at $x = 0$ in order to evaluate the trade-off between the run and non-run equilibria.

3 The Post-deposit Game

The post-deposit game starts in period 1 after deposits have been made. It is a game among the patient depositors on whether to withdraw early or to wait to withdraw in period 2. Impatient depositors always choose early withdrawals (independent of what other depositors choose). For a patient depositor, his choice *does* depend on his beliefs about the behavior of the other patient depositors.

Let $c^{run\_proof} = 1$ and $c^{IC} = \frac{R}{(1-\lambda)+\lambda R}$, where c^{run_proof} is the level of c beyond which the bank's assets will be depleted in period 1 if all depositors withdraw early. A patient depositor, anticipating that other patient depositors choose late withdrawal, prefers late withdrawal if and only if $c \leq c^{IC}$. The superscript IC stands for "incentive compatible". Therefore, for $c \in (c^{run\_proof}, c^{IC}]$, there are two equilibria in the post-deposit game: one run and the other non-run.[4] However, for $c \in [0, c^{run\_proof}]$ or $c \in (c^{IC}, \frac{1}{\lambda}]$, the equilibrium is unique. In the former case, the unique equilibrium is non-run. In the latter case, the unique equilibrium is the run equilibrium. The 3 cases are graphed below:

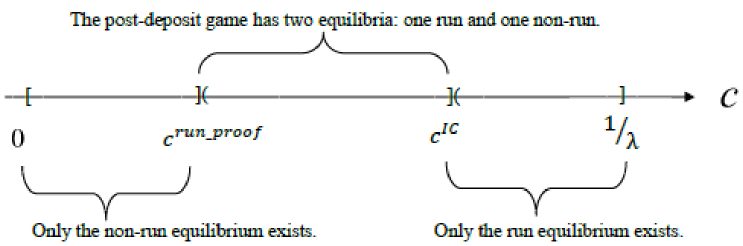

Figure 1: The 3 Cases in the Post-Deposit Game

If $c > c^{run\_proof} = 1$, a patient depositor knows that there will be resources remaining in the bank in Period 2 even if everyone else withdraws early. Therefore, a run equilibrium exists if and only if $c > 1$ since a patient depositor strictly prefers early withdrawal if he expects everyone else withdraws early. For $c \leq c^{IC}$, a patient depositor *weakly* prefers late withdrawal

[4] We consider here only symmetric equilibria. The run equilibrium is definded as an equilibrium in which all patient depositors choose early withdrawal. The non-run equilibrium is defined as one in which no patient depositor chooses early withdrawal.

if he expects the other patient depositors make late withdrawals. Therefore, a non-run equilibrium exists if and only if $c \leq c^{IC}$.

We will next move backward to period 0 in order to analyze the pre-deposit game.

4 The Pre-deposit Game

The pre-deposit game is a game between the bank and the consumer. The bank chooses the deposit contract c and the consumer chooses whether to deposit or not. Since the banking industry is free-entry, the deposit contract chosen by the bank is the one which maximizes the depositor's expected utility. Since the post-deposit game might involve multi-equilibria, we need to specify the probability of each possible outcome.

4.1 Run Probability: Rational Beliefs

At the beginning of period 1, each depositor publicly observes a random variable σ, which is uniformly distributed on $[0, 1]$. We define rational beliefs about the run probability by:

(1) If a run outcome and non-run outcome co-exist in the post-deposit game, each depositor *believes* that all other patient depositors will choose early-withdrawal if and only if the realized value of σ is no larger than $s^*(c) \in (0, 1)$.[5] These are rational beliefs in sunspot-driven, panic-based runs.

(2) If there is a unique outcome – either a run or a non-run – in the post-deposit game, each depositor believes that all other depositors choose the correct (non-stochastic) strategy independent of σ.

To summarize, the rational run probability belief $s(c)$ is given by

$$s(c) = \begin{cases} 0, \text{ if } c \in [0, c^{run\_proof}] \\ s^*(c), \text{ if } c \in (c^{run\_proof}, c^{IC}] \\ 1, \text{ if } c \in (c^{IC}, 1/\lambda]. \end{cases}$$

[5] When we have multiple exquilibria, we assume that each equilibrium occurs with strictly positive probability. That is why $s^*(c)$ cannot be 0 or 1.

The simplest case would be $s^*(c) = s_1 \in (0,1)$. See Figure 2. This is the special 1-step function employed by Peck and Shell [11]. The red line segments summarize depositor behavior assumed in Peck and Shell [11].

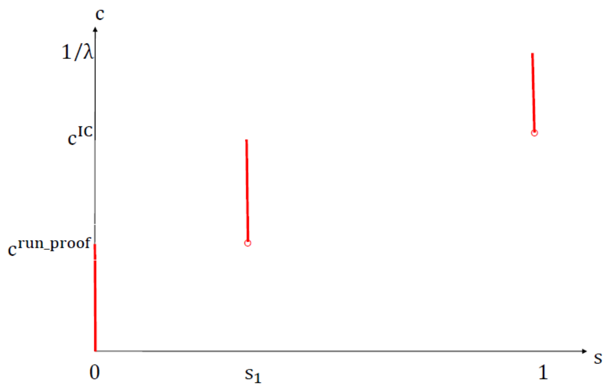

Figure 2

4.2 The Optimal Contract

For any given run probability s, the optimal contract in the pre-deposit game is c^* which maximizes the depositors' expected utility:

$$c^* = \arg\max_c W(c, s(c)),$$

where

$$W(c, s(c)) = \begin{cases} \lambda u(c) + (1-\lambda)u(\frac{1-\lambda c}{1-\lambda}R), \text{ if } c \in [0, c^{run\_proof}] \\ \\ s^*(c)[\frac{1}{c}u(c)] + (1 - s^*(c))[\lambda u(c) + (1-\lambda)u(\frac{1-\lambda c}{1-\lambda}R)], \\ \text{if } c \in (c^{run\_proof}, c^{IC}] \end{cases}$$

Note that, when c^* is chosen by the bank, its effect on the run probability is considered. This is not what a myopic bank would do.

As a comparison, we first describe the behavior of a myopic bank, mostly to introduce our notation. (When we analyze equilibrium, we will assume that the bank is not myopic.) For any given run probability s, the optimal contract chosen by the myopic bank in the pre-deposit game is $c^*(s)$ which maximizes the depositors' ex-ante utility:

$$c^*(s) = \arg\max_c W(c; s)$$

where

$$W(c;s) = \begin{cases} \lambda u(c) + (1-\lambda)u(\frac{1-\lambda c}{1-\lambda}R), \text{ if } c \in [0, c^{run\_proof}] \\ \\ s[\frac{1}{c}u(c)] + (1-s)[\lambda u(c) + (1-\lambda)u(\frac{1-\lambda c}{1-\lambda}R)] \\ \text{if } c \in (c^{run\_proof}, c^{IC}] \\ \\ \frac{1}{c}u(c), \text{ if } c \in (c^{IC}, 1/\lambda]. \end{cases}$$

We see that $c^*(0) = \frac{R^{1-\frac{1}{\gamma}} - (1-\lambda)(1-R^{-\frac{1}{\gamma}})}{\lambda R^{1-\frac{1}{\gamma}} + (1-\lambda)}$ corresponds to the "unconstrained efficient allocation" denoted by c^{UE}. Since $c^{IC} = \frac{R}{\lambda R + (1-\lambda)}$, we have that $c^{UE} < c^{IC}$: the IC constraint never binds in the present model based on the original DD environment.[6] Define $s_0(c)$ as the maximum value of s beyond which it is no longer optimal to tolerate runs under the contract c.

Example 1 $u(c) = \frac{(c+1)^{1-\theta}}{1-\theta} + 1$, *where* $\theta = 3$. $R = 2, \lambda = 0.3$. *In this example, we have* $c^{run\_proof} = 1$, $c^{IC} = 1.538$ *and* $c^{UE} = 1.227$. *We plot* $s_0(c)$ *and* $c^*(s)$ *in Figure 3. Let* $\overline{s_0} = \max_c(s_0(c))$. *We see that* s_1 *is an off-equilibrium belief if* $s_1 \geq s_0$; $s_1 \in (0, s_0)$ *is an equilibrium belief. The corresponding equilibrium contract is* $c^*(s_1)$.

[6] If the impatient consumer's utility function is different from the patient consumer's besides just the goods they value, the IC constraint binds for some parameters; see [12].

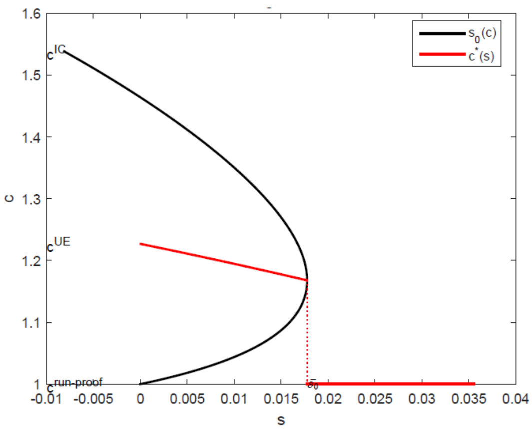

Figure 3

5 Equilibrium in the Pre-deposit Game

The REE in the pre-deposit game is denoted as $(\widehat{s}, c^*)$, which is an equilibrium value of the pair of the depositor run-probability and the optimal contract chosen by the bank.

5.1 s*(c) is a single-step function

$$s(c) = \begin{cases} 0, & \text{if } c \in [0, c^{run\_proof}] \\ s_1, & \text{if } c \in (c^{run\_proof}, c^{IC}] \\ 1, & \text{if } c \in (c^{IC}, 1/\lambda]. \end{cases}$$

Proposition 1 *If $s_1 \in (0, \overline{s_0})$, the unique REE is $(s_1, c^*(s_1))$. If $s_1 > \overline{s_0}$, the unique REE is $(0, c^{run-proof})$. If $s_1 = \overline{s_0}$, there are two equilibria: $(\overline{s_0}, c^*(\overline{s_0}))$ and $(0, c^{run-proof})$.*

5.2 $s^*(c)$ is a multiple-step function

Let $s^*(c) = \begin{cases} 0, \text{ if } c \in [0, c^{run\_proof}] \\ \\ s_1, \text{ if } c \in (c^{run\_proof}, c^1] \\ \\ s_2, \text{ if } c \in (c^1, c^{IC}] \\ \\ 1, \text{ if } c \in (c^{IC}, 1/\lambda], \end{cases}$
where $0 < s_1 < s_2 < 1$.

Example 2 *Use the parameter values from in Example 1. Let $s^*(c)$ be a multiple-step function with $s_1 = 0.0053$, $s_2 = 0.0107$ and c^1 =1.083. Then we have that s_1 and s_2 are equilibrium run beliefs. The corresponding equilibrium contract is c^1 =1.083, c^2 =1.192.*

If $s^*(c)$ is a single-step function with $s^*(c) = s_1$, the ex-ante welfare tolerating runs, $W(s_1)$, as a function of c is the blue curve in the Figure 4. Similarly, if $s^*(c)$ is a single-step function with $s^*(c) = s_2$, $W(s_2)$, is the black curve in Figure 4.

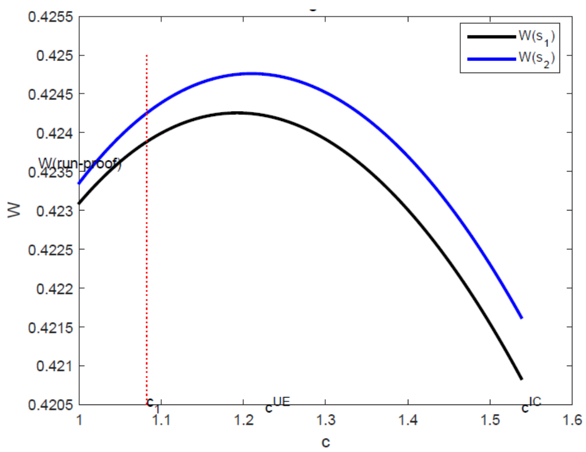

Figure 4

Therefore, if $s^*(c)$ jumps from s_1 to s_2 at c^1, the ex-ante contract (see the solid curve in Figure 5) tolerates runs, is discontinues at c^1, and is composed of two pieces: the blue curve up to c^1 and the black curve beyond c^1. There are two optimal, run-tolerating contracts corresponding to the two probability beliefs: $c = c^1$ =1.083 for s_1 and $c = c^2$ =1.192 for s_2. Each contract delivers a higher level of welfare than the run-proof contract ($W(c^{run\_proof})$). Each delivers expected utility $W = 0.4243$. The second contract is riskier ($s_2 > s_1$), but the extra first-period withdrawal ($c^2 > c^1$) exactly compensates for the increased risk. The contract c_1 is the boundary solution to the bank's maximization problem, while c_2 is the interior solution, in which the bank's first-order condition is satisfied with equality.

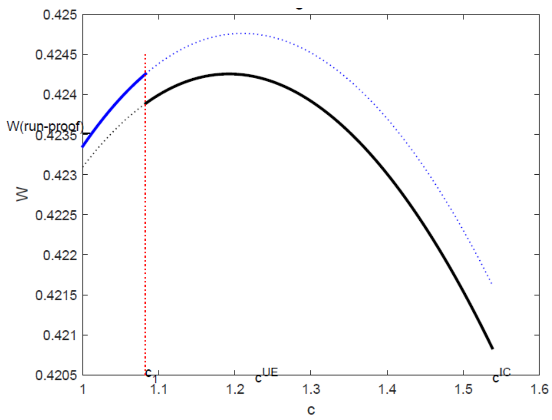

Figure 5

5.3 s*(c) is continuous and strictly-increasing in c

From the definition of expected utility,

$$W(c, s(c)) = \begin{cases} \lambda u(c) + (1-\lambda)u(\frac{1-\lambda c}{1-\lambda}R), \text{ if } c \in [0, c^{run\_proof}] \\ \\ s^*(c)[\frac{1}{c}u(c)] + (1-s^*(c))[\lambda u(c) + (1-\lambda)u(\frac{1-\lambda c}{1-\lambda}R)], \\ \text{if } c \in (c^{run\_proof}, c^{IC}], \end{cases}$$

we know that $W(c, s(c))$ is continuous in c when runs are tolerated, guaranteeing uniqueness of equilibrium.

$W(c, s(c))$ is not always smooth or concave. Multiple equilibria can exist. The next example establishes that there can be multiple equilibria when $s^*(c)$ and hence $W(c, s(c))$ have kinks. We construct the example so that $s^*(c)$ is a continuous function which "mimics" the multiple-step function in Example

2. The two equilibria in Example 2 continue to be equilibria in Example 3.

Example 3 $$s^*(c) = \begin{cases} 0.0048 + 0.0064(c - 1.0003), & \textit{if } c \in (c^{run\_proof}, 1.083] \\ 0.0053 + 2.4072(c - 1.083), & \textit{if } c \in (1.083, 1.085] \\ 0.0106 + 0.0005(c - 1.085), & \textit{if } c \in (1.085, 1.192] \\ 0.0107 + 0.0031(c - 1.192), & \textit{if } c \in (1.192, c^{IC}] \end{cases}$$

We plot $s^*(c)$ in Figure 6. We also plot the step function from Example 2 for comparison. We plot $W(c, s(c))$ in Figure 7. The two equilibria are (s_1, c^1) and (s_2, c^2) as in Example 2.

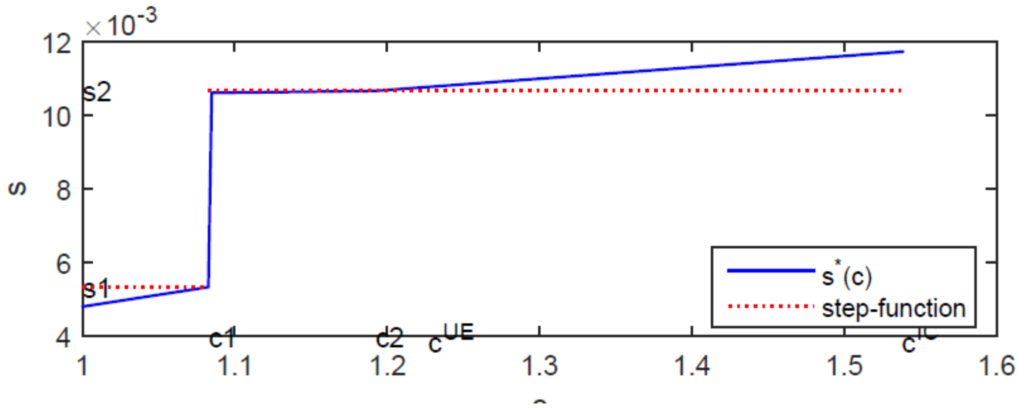

Figure 6

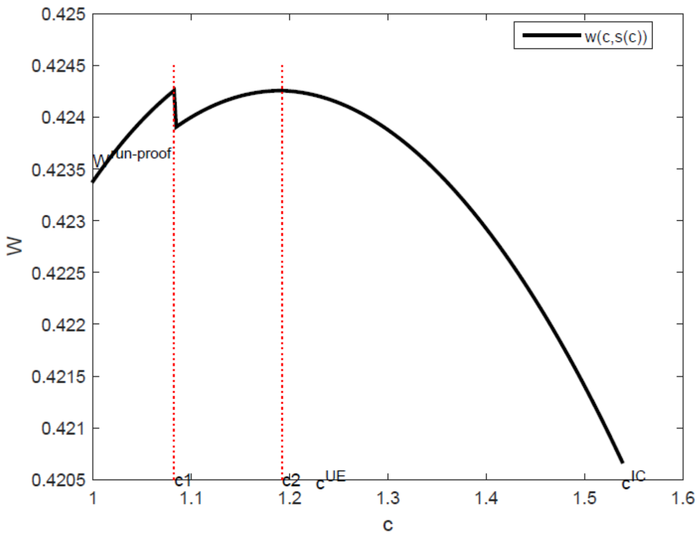

Figure 7

6 Summary and Concluding Remarks

We celebrate Diamond-Dybvig [3] and its precursor Bryant [2]. These are breakthrough papers. Because of asymmetric information — the depositor knows his type (patient or impatient) while others including the bank teller do not know the depositor's type — a contract, or in the words of Neil Wallace [13] a mechanism — plays a useful role in the absence of a Walrasian contingent market for liquidity. Hence DD have given us a theoretical foundation for financial intermediation. The DD model also offers us a spring-board for understanding other phenomena in financial economics, monetary economics, and macroeconomics.

DD solved for the best deposit contract for a situation in which depositors were assumed to avoid running. While this contract supports a non-run outcome (the unrestricted efficient allocation), it does not do so uniquely. There is another outcome, the run outcome. Are these outcomes equilibria? Would anyone deposit in a bank with a 100% chance of experiencing a run? Would a bank offer a deposit contract which would surely lead to a run? These questions lead DD and many distinguished followers to re-design the mechanism to prevent runs. In the case of DD, deposit insurance was introduced to avoid runs in the post-deposit game.

An alternative approach is to allow for the possibility of runs. The optimal contract might tolerate runs with small probabilities of occurrence. Pedestrians cross streets even when it is not perfectly safe to do so. Engineers do not build bridges to last forever.[7] Peck and Shell [11] introduced the pre-deposit game, in which consumers decide whether or not to deposit in the bank, while the bank designs the banking contract with consumer choice in mind. In [11], the pre-deposit game is based on a more complicated environment than that of DD (in an attempt to dodge potential criticisms based on a literature exemplified by Green and Lin [9], in which richer mechanisms are employed to rule out runs in the post-deposit game.) The pre-deposit game has by now been employed in several bank runs papers and is applied here to the classic DD environment.

In the present paper, we extend the DD analysis to study off-equilibrium effects of feasible (but not necessarily optimal) contracts. The set of feasible contracts is divided into 3 subsets: A most conservative contracts (small c) results in a unique allocation which is non-run. A least conservative contract (large c) results in a unique allocation which is run. In other cases, the contract leads to 2 outcomes: one is non-run, the other is run. This global analysis was first applied in Shell and Zhang [12] to a rather different 2-depositor environment, but the results are similar when applied to the classic DD environment.

In the present paper, a depositor or potential depositor coordinates his belief about the behavior of other depositors by observing a sunspot-signal noiselessly observed by all depositors. Asymmetric sunspots would have been better; see Gu [8] and [1]. In the present paper, there is no aggregate uncertainty about the fundamentals. We believe that this could be included without too much pain.

In [11], the expectations of the depositors are from a special family: the probability s of a run given a deposit contract c, $s(c)$, is constant in c over the full range of feasible c. Furthermore, in this example the sunspot equilibrium

[7]Furthermore, bank runs have occurred in history. It would be helpful to be able to explain their occurrence.

is a mere randomization over the 2 outcomes from the post-deposit game. In [12], we showed that in the PS numerical example, incentive compatibility is binding, and that when IC in not binding the sunspot equilibrium is not a mere randomization over post-deposit-game outcomes. We extend the result [12] to the present paper. Also, as in [12], when the IC does not bind in the present model the bank contract c is strictly declining in the depositor run probability s.

In the present paper, we follow the lead from Ennis and Keister [4] and [5] who have analyzed more general belief mechanisms in two beautiful papers on coordination games. We extend the 1-step belief function of [11] to a multi-step belief function. Under these beliefs there is an equilibrium in the pre-deposit game between the potential depositors and the bank. This equilibrium is not necessarily unique. We show for the DD model that if depositors believe that s is a continuous, smooth, and increasing function of the contract c, then the equilibrium is unique. Smoothness is essential to the result. We present an example of a continuous, increasing function $s(c)$ with a kink which results in existence of equilibrium, but not uniqueness.

We have left out from the present paper, more than we put in. We apologize. We have not taken sequential service seriously; see Wallace [13]. We have not allowed for partial suspension of convertibility: see Wallace [14]. We have not allowed for imperfect commitment; see Ennis and Keister [6]. And much more.

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