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On sunspots, bank runs, and Glass–Steagall

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We analyze the pre-deposit game in a two-depositor banking model. The Glass–Steagall bank is assumed to be restricted to holding only liquid assets. Depositors tolerate a panic-based run if its probability of occurrence s is small. How s affects the allocation of assets depends on the incentive compatibility constraint (ICC). When the ICC is not binding, the sunspot allocation is not a mere randomization over the run and non-run outcomes under the so-called "optimal contract." We offer this paper as a contribution to both the literature on banking and financial fragility and also the broader literature on sunspot equilibrium.

Key words bank run, deposit contract, Glass–Steagall banking, illiquid asset, liquid asset, merchant banking, pre-deposit game, post-deposit game, run probability, sunspot

JEL classification G21, E44

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1 Introduction

We analyze a banking model based on Peck and Shell (2010). As in Wallace (1996), there are two investment assets: one liquid and the other illiquid. Two financial systems are compared. In the *separated financial system* there are two separate institutions. One holds only the liquid asset. This might be thought of as a Glass–Steagall bank (GSB), or a narrow bank. The other financial institution holds only the illiquid asset. It is like a stock brokerage or mutual fund. In the *unified financial system*, or consolidated system, one institution holds the two assets. This might be thought of as a merchant bank, or merely a post-Glass–Steagall, pre-Dodd–Frank, modern bank.

As in Peck and Shell (2010), we introduce intrinsic aggregate uncertainty by assuming that the realized fraction of impatient consumers is itself stochastic. Peck and Shell (2010) assume that there is a continuum of consumers. We assume that there are only a finite number of consumers.

In the post-deposit game, the unified financial system is immune from sunspot-driven runs, but for some parameters the unified system runs out of cash when the realized fraction of impatient consumers is high. The GSB is always susceptible to a sunspot-driven run. The run probability s is an exogenous parameter summarizing the "mood" of the financial sector. If the contract permits a run, then the probability of the run is s. Of course, if the contract does not permit a run, the probability of a run is 0. How does the run probability s affect the optimal contract? For small s, runs are tolerated. In some cases, the contract becomes (as one would expect) strictly more conservative as s increases.

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We dedicate this paper to Jess Benhabib and Roger Farmer, the fathers of empirical sunspot modeling of the macro-economy.

In other cases, the contract is locally constant in s. In the first cases (the intuitive ones) the incentive constraint is not binding. In the second cases the incentive constraint is binding.

Our paper is a contribution to the banking and related finance literatures. It is also a contribution to the general sunspots literature, making it appropriate for inclusion in an issue of the *International Journal of Economic Theory* in honor of Roger Farmer. In the present model, the sunspot equilibrium allocation is sometimes a mere randomization over the allocations from the post-deposit game. Otherwise it is an entirely different animal. This second is like the sunspot equilibria that are not mere randomizations over the competitive equilibria from the corresponding certainty economy.

2 The model

2.1 Preferences and technologies

As in Shell and Zhang (2018), there are two consumers and three periods: 0, 1 and 2. In period 0, each consumer is endowed with y units of the consumption good. There are no endowments in periods 1 and 2. In period 1, either (1) one consumer becomes patient and the other becomes impatient, or (2) both consumer are patient. The probabilities of the two cases are q and $1 - q$, respectively. Thus, the aggregate number of patient depositors is stochastic.[1]

Following Peck and Shell (2010), an impatient consumer has an "indivisible" consumption opportunity[2] in period 1, yielding incremental utility of $\overline{u}$ for one unit of consumption in period 1.[3] If the consumption opportunity goes unfulfilled in period 1, these consumers face a discounted consumption opportunity in period 2, yielding incremental utility of $\beta\overline{u}$ for one unit of consumption in period 2, where the scalar β is less than unity. For a patient consumer, the indivisible consumption opportunity arrives in period 2. Beyond these urgent consumption opportunities, both types of consumers derive utility from left-over consumption in period 2 with the strictly concave utility $u()$ for the left-over balances. Thus, impatient and patient consumers, respectively, have the reduced-form utility functions

$$U_I(C_I^1, C_I^2) = \begin{cases} \overline{u} + u(C_I^1 + C_I^2 - 1), & \text{if } C_I^1 \geq 1, \\ \beta\overline{u} + u(C_I^1 + C_I^2 - 1), & \text{if } C_I^1 < 1, \end{cases} \tag{1}$$

and

$$U_P(C_P^1, C_P^2) = \overline{u} + u(C_P^1 + C_P^2 - 1), \tag{2}$$

where $C_j^t \geq 0$ is the total withdrawal of a type-j depositor from the bank in period t. I stands for impatient depositor and P stands for patient depositor.

There are two constant-returns-to-scale technologies: an illiquid, higher-yield technology A, and a liquid, lower-yield technology B. For the illiquid technology, investing one unit of period-0

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[1] As in Peck and Shell (2010), nature first determines how many consumers are patient in the economy. Conditional on there being a patient consumer, the probability of each consumer being patient is $\frac{1}{2}$. The *ex ante* probability that a given consumer becomes impatient is $\frac{q}{2}$.

[2] This "indivisibility" is meant to capture the idea that if payment is not made at par the purchase is lost (see Peck and Shell 2010).

[3] Our notation is mostly, but not entirely, from Peck and Shell (2010).

consumption yields R_A units of consumption if harvested in period 2 and nothing if harvested in period 1. For the liquid technology, investing one unit of period-0 consumption yields R_B units of consumption if harvested in period 2 and one unit of consumption if harvested in period 1. We assume that $1 < R_B < R_A$ holds. We also assume that individuals do not have direct access to the two productive technologies.[4] Individuals can costlessly store the consumption good.

2.2 The space of deposit contracts

In period 0, the bank designs the demand-deposit contract. Following the bank runs literature, we assume that the competitive bank maximizes the *ex ante* expected utilities of depositors. At the beginning of period 1, each depositor learns his type and observes a sunspot variable δ distributed uniformly on $[0, 1]$. Then, the depositors decide whether to arrive at the bank in period 1 or in period 2. Depositors who choose period 1 arrive in random order.

Let γ denote the fraction of a depositor's endowment invested in technology B. The aggregate resources available in period 1 devoted to investment in the liquid asset are thus $2y\gamma$. Sequential service constraints are part of the physical environment (see Wallace 1988). Let $c^1(z)$ be the withdrawal in period 1 for a depositor with position z in the queue, where $z \in \{1, 2\}$. Given the indivisible consumption opportunity, we know that in the optimal contract, $c^1(z)$ will take either the value of 1 or 0. Furthermore, as in Peck and Shell (2010), we restrict attention to environments in which it is beneficial to provide for consumption opportunities whenever the resources are available.[5]

$$c^1(z) = \begin{cases} 1, & \text{if } 2y\gamma \geq z, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Let α denote the number of depositors who have made a withdrawal in period 1. We have

$$\alpha \in \{0, 1, 2\} \quad \text{and} \quad \alpha \leq 2y\gamma.$$

In period 2, the bank chooses how to divide its remaining resources from technology B between those who have withdrawn in period 1 and those who have not as functions of α, denoted respectively by $c_I^2(\alpha)$ and $c_P^2(\alpha)$.[6] For the deposit contract to be feasible, all remaining resources must be distributed in period 2. The resource constraint (RC) is given by

$$\alpha c_I^2(\alpha) + (2 - \alpha)c_P^2(\alpha) = \left[2y\gamma - \sum_{z=1}^{\alpha} c^1(z)\right] R_B. \tag{4}$$

The space of deposit contracts, or mechanisms, for the unified financial system, M^U, is given by

$$M^U = \{\gamma, c_I^2(\alpha), c_P^2(\alpha) \mid \text{Equation (4) holds}\}.$$

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[4] Allowing such direct access opens the possibility of disintermediation, in which direct investment is superior to depositing in the GSB bank. We plan to investigate disintermediation in future research.

[5] That is, we assume $\overline{u}$ is sufficiently large.

[6] That is, as in Peck and Shell (2010), a consumer who receives $c_I^2(\alpha)$ from the liquid asset investment receives a total withdrawal in period 2 of $C_I^2(\alpha) = c_I^2(\alpha) + (1 - \gamma)R_A y$. Similarly, a consumer who receives $c_p^2(\alpha)$ from the liquid asset investment receives a total withdrawal in period 2 of $C_P^2(\alpha) = c_P^2(\alpha) + (1 - \gamma)R_A y$. As in Peck and Shell (2010), it is assumed that parameters are such that the inequalities $C_I^2(\alpha) \geq 0$ and $C_P^2(\alpha) \geq 0$ never bind.

The unified bank can access the illiquid asset in period 2, and therefore, $c_I^2(\alpha)$ and $c_P^2(\alpha)$ can be negative. For the separated financial system, neither $c_I^2(\alpha)$ nor $c_P^2(\alpha)$ can be negative since the GSB does not have access to the illiquid asset. The space of deposit contracts or mechanisms for the separated financial system, M^S, is given by

$$M^S = \{\gamma, c_I^2(\alpha) \geq 0, c_P^2(\alpha) \geq 0 \mid \text{Equation (4) holds}\}.$$

It follows that $c_I^2(0)$ is not relevant. When $\alpha = 0$, no one has withdrawn in period 1. Similarly, $c_P^2(2)$ is not relevant. Therefore, both $c_P^2(0)$ and $c_I^2(2)$ can be solved directly from the resource constraint as a function of γ. What we are left to determine in the space of deposit contracts is $\{\gamma, c_I^2(1), c_P^2(1)\}$. We will show later that $c_I^2(1)$ and $c_P^2(1)$ will be determined by smoothing of left-over consumption and the incentive compatibility constraint (ICC). The contract is characterized by the fraction γ.

3 The post-deposit game

A non-run equilibrium in the *post-deposit* game is a Bayes–Nash equilibrium (BNE) in which only impatient depositors arrive and withdraw in period 1. The ex-ante welfare at the non-run equilibrium, $W^{\text{non-run}}$, depends on the total liquidity available to the bank in period 1, namely $2y\gamma$. For $\gamma \in [0, \frac{1}{2y})$, no depositor can withdraw in period 1. Therefore, we have

$$\begin{aligned} W^{\text{non-run}} &= q\{\overline{u} + \beta\overline{u} + 2u[(1-\gamma)yR_A + \gamma yR_B - 1]\} \\ &\quad + (1-q)\{2\overline{u} + 2u[(1-\gamma)yR_A + \gamma yR_B - 1]\}. \end{aligned} \tag{5}$$

In this case, even though a depositor becomes impatient, he has to defer his indivisible consumption opportunity. For $\gamma \in [\frac{1}{2y}, 1]$, the impatient depositor (if he exists) does not have to wait for the next period. Therefore, we have

$$\begin{aligned} W^{\text{non-run}} &= q\{2\overline{u} + u[(1-\gamma)yR_A + c_I^2(1)] + u[(1-\gamma)yR_A + c_P^2(1) - 1]\} \\ &\quad + (1-q)\{2\overline{u} + 2u[(1-\gamma)yR_A + \gamma yR_B - 1]\}. \end{aligned} \tag{6}$$

Since depositors' types are private information, non-run equilibrium requires that the ICC holds. That is, a patient depositor weakly prefers period-2 arrival to period-1 arrival, when he assumes that the other patient depositor chooses period-2 arrival. Since a patient depositor's consumption opportunity is in period 2 and consumption can be costlessly stored, his comparison between arrivals in the two periods only depends on the left-over consumption. For $\gamma \in [0, \frac{1}{2y})$, no one can withdraw in period 1, and therefore the ICC is irrelevant. For $\gamma \in [\frac{1}{2y}, 1]$, at most one depositor can withdraw in period 1, and the ICC is given by

$$\begin{aligned} &\frac{q/2}{q/2 + (1-q)} u[(1-\gamma)yR_A + c_P^2(1) - 1] + \frac{(1-q)}{q/2 + (1-q)} u[(1-\gamma)yR_A + c_P^2(0) - 1] \\ &\geq \frac{q/2}{q/2 + (1-q)} \left\{ \frac{1}{2} u[(1-\gamma)yR_A + c_I^2(1)] + \frac{1}{2} u[(1-\gamma)yR_A + c_P^2(1) - 1] \right\} \\ &\quad + \frac{(1-q)}{q/2 + (1-q)} u[(1-\gamma)yR_A + c_I^2(1)]. \end{aligned} \tag{7}$$

The left- and right-hand sides of Inequality (7) are a patient depositor's expected utilities from the left-over consumption if he arrives in period 2 and 1 respectively, when he assumes that the other patient depositor chooses period-2 arrival. If a patient depositor is second in the queue, he knows that there will be no withdrawals in period 1 after him, and therefore he does not withdraw in period 1.[7] Other parts of the ICC are verified in like manner.

3.1 The unified financial system

The non-run optimal contract (NROC) for the unified bank is a contract belonging to M^U which maximizes $W^{\text{non-run}}$ subject to the ICC. Several observations follow. First, the ICC never binds in the unified system. For $\gamma \in [0, \frac{1}{2y})$, the ICC is not relevant. For $\gamma \in [\frac{1}{2y}, 1]$, the incentive to smooth left-over consumption across depositors and the RC imply that

$$c_I^2(1) = c_P^2(1) - 1 = \gamma y R_B - \frac{R_B + 1}{2}.$$

From the RC, we also have

$$c_P^2(0) - 1 = \gamma y R_B - 1.$$

Hence the ICC does not bind for $\gamma \in [\frac{1}{2y}, 1]$. Furthermore, since the ICC does not bind and complete consumption smoothing is available, the incentive to economize on the liquid asset is the only concern when choosing γ in both $[0, \frac{1}{2y})$ and $[\frac{1}{2y}, 1]$. Therefore, to find the NROC for the unified system, we merely need to compare the values of $W^{\text{non-run}}$ when $\gamma = 0$ versus when $\gamma = \frac{1}{2y}$.

Proposition 1 *If $\overline{u}$ is larger than the threshold $\overline{u}_0$, the unified bank never runs out of liquidity in period 1 (i.e. $\gamma = \frac{1}{2y}$). Otherwise, the unified bank holds only the illiquid asset (i.e. $\gamma = 0$). The threshold $\overline{u}_0$ is equal to*

$$\frac{u(yR_A - 1) - qu(yR_A - \frac{R_A+1}{2}) - (1-q)u(yR_A - \frac{R_A - R_B}{2} - 1)}{\frac{q}{2}(1-\beta)}$$

All proofs are in the Appendix.

In Peck and Shell (2010), the unified system *always* runs out of cash when the realized fraction of impatient consumers is high. This difference is due to the fact that in Peck and Shell (2010) there is a continuum of depositors and the fraction of impatient depositors, α, is continuously distributed with support $[0, \overline{\alpha}]$. Therefore, the probability of running out of liquidity increases continuously as the liquidity is lowered below $\overline{\alpha}$. Hence if liquidity is smaller than but sufficiently close to $\overline{\alpha}$, the *expected* cost of running out of liquidity diminishes while the benefit due to higher return from the illiquid asset does not. However, for a model like ours with a finite number of depositors, that probability increases abruptly, from 0 to q, as γ is lowered below $\frac{1}{2y}$. Therefore, the expected cost of running out of liquidity cannot be made arbitrarily small and is fixed at $q(1-\beta)\overline{u}$, which is the expected utility loss due to the postponement of the consumption opportunity by the impatient

[7] As in Peck and Shell (2010), we assume that when making the withdrawal decision, a depositor in the queue knows what the guy ahead of him in line has done, and a depositor who does not withdraw in period 1 can withdraw in period 2 without prejudice. This assumption ensures that for no parameter values will there be strategic substitutability. Compare this with Shell and Zhang (2018), in which there is strategic substitutability for some parameter values.

depositor. Hence, depending on the parameters, the NROC might entail that the unified bank never runs out of liquidity. In fact, this is the case when $\overline{u} > \overline{u}_0$. If the consumption opportunity is more important (i.e. larger $\overline{u}$), or the discounting factor is smaller (i.e. smaller β), or it is more likely to have an impatient depositor (i.e. q is higher), the expected cost of running out of liquidity is larger. That is why the threshold level $\overline{u}_0$ is negatively related to $(1-\beta)$ and q.

If $\overline{u} \leq \overline{u}_0$, the bank is redundant since in autarky the depositors can do equally well by holding only the illiquid asset. Hence for the remainder of the analysis, we focus on the case in which $\overline{u}$ is sufficiently large such that we only need to consider $\gamma \in [\frac{1}{2y}, 1]$.

A run equilibrium in the post-deposit game is defined as a BNE in which both depositors, regardless of types, arrive in period 1. A run equilibrium exists if and only if a patient depositor strictly prefers arriving in period 1. That is,

$$\frac{1}{2}u[(1-\gamma)yR_A + c_I^2(1)] + \frac{1}{2}u[(1-\gamma)yR_A + c_P^2(1) - 1]$$
$$> u[(1-\gamma)yR_A + c_P^2(1) - 1]. \tag{8}$$

As with the ICC, Inequality (8) only involves utilities from left-over consumption. The left-hand side (and right-hand side) of Inequality (8) is a patient depositor's expected utility from left-over consumption if he arrives in period 1 (and period 2), while assuming the other patient depositor (if he exists) chooses period-1 arrival. Obviously, Inequality (8) can be reduced to

$$c_I^2(1) > c_P^2(1) - 1. \tag{9}$$

Since the NROC entails complete left-over consumption smoothing, $c_I^2(1) = c_P^2(1) - 1$, the unified banking system is immune to panic-based runs.

3.2 The separated financial system

The NROC for the separated banking system is a contract belonging to M^S which maximizes $W^{\text{non-run}}$ subject to the ICC. Unlike the unified system, we must have non-negative $c_I^2(1)$ in the separated system since the GSB does not have access to the illiquid asset. Therefore, the incentive to smooth left-over consumption across depositors is restricted by the non-negativity constraint if the bank does not have sufficient liquidity. To be specific, for $\gamma \in [\frac{1}{2y}, \frac{1}{2y} + \frac{1}{2yR_B}]$, the non-negativity constraint binds and

$$c_P^2(1) - 1 < c_I^2(1) = 0.$$

For $\gamma \in (\frac{1}{2y} + \frac{1}{2yR_B}, 1]$, the non-negativity constraint does not bind so there is complete consumption smoothing. Combining this with the resource constraint, we have

$$c_I^2(1) = \max\left\{0, \gamma y R_B - \frac{R_B+1}{2}\right\}$$

and

$$c_P^2(1) - 1 = \min\left\{(2\gamma y - 1)R_B - 1, \gamma y R_B - \frac{R_B+1}{2}\right\}.$$

If the bank has sufficient liquidity so that left-over consumption is completely smoothed across depositors, the ICC does not bind (as in the unified system). Furthermore, from condition (8), we know that the non-run equilibrium is the unique equilibrium. However, if the GSB does not have sufficient liquidity, the ICC might bind depending on the parameters. Furthermore, from condition (8), we know that a run equilibrium also exists. To summarize, we have the following lemma.

Lemma 1 *If the GSB has sufficient liquidity (i.e. $\gamma \geq \frac{1}{2y} + \frac{1}{2yR_B}$), the ICC does not bind and non-run is the unique equilibrium in the post-deposit game. Otherwise, the ICC might bind depending on the parameters and we also have a run equilibrium in the post-deposit game.*

Let $\gamma^{\text{run-proof}}$ denote the value of $\frac{1}{2y} + \frac{1}{2yR_B}$. From Lemma 1 we know that a run equilibrium exists if and only if $\gamma < \gamma^{\text{run-proof}}$. The next proposition shows that the NROC in the separated system also has a run equilibrium. This result is the same as the result in Peck and Shell (2010), where there is a continuum of depositors.

Proposition 2 *Let $\widehat{\gamma}$ denote the fraction of the liquid asset investment specified by the NROC. We have $\widehat{\gamma} < \gamma^{\text{run}-\text{proof}}$.*

The NROC only considers the non-run equilibrium, so the benefit from eliminating runs is not included in this calculation. Therefore, the only benefit of increasing γ to $\gamma^{\text{run-proof}}$ is smoothing of left-over consumption when there is an impatient depositor. The cost of doing this is the lower return compared to the illiquid asset. But when γ is sufficiently close to $\gamma^{\text{run-proof}}$, left-over consumption is sufficiently close to complete smoothing. So the benefit of raising γ diminishes while the cost does not.

4 The pre-deposit game

Since the NROC in the unified system is immune to runs, the optimal contract in the pre-deposit game is the same as the NROC from the post-deposit game. So in this section, we only analyze the pre-deposit game for the separated system. In Section 3, we showed that in the post-deposit game of the separated system, we have both a non-run equilibrium and a run equilibrium for any $\gamma \in [\frac{1}{2y}, \gamma^{\text{run-proof}})$. For any $\gamma \in [\gamma^{\text{run-proof}}, 1]$, we have a unique non-run equilibrium. We assume that bank runs are sunspot-driven and the run probability is s. Hence whether a run equilibrium exists in the *pre-deposit* game depends on whether the optimal contract for the pre-deposit game, denoted by $\gamma^*(s)$, belongs to the set $[\frac{1}{2y}, \gamma^{\text{run-proof}})$.

Denote the *ex ante* welfare by $W(\gamma; s)$. We have

$$W(\gamma; s) = \begin{cases} sW^{\text{run}}(\gamma) + (1-s)W^{\text{non-run}}(\gamma), & \text{if } \gamma \in [\frac{1}{2y}, \gamma^{\text{run-proof}}], \\ W^{\text{non-run}}(\gamma), & \text{if } \gamma \in [\gamma^{\text{run-proof}}, 1], \end{cases} \tag{10}$$

where

$$W^{\text{run}}(\gamma) = q\left(\frac{\overline{u} + \beta\overline{u}}{2} + \frac{2\overline{u}}{2}\right) + (1-q)(2\overline{u})$$
$$+ u[(1-\gamma)yR_A + c_I^2(1)] + u[(1-\gamma)yR_A + c_P^2(1) - 1]. \tag{11}$$

The *ex ante* welfare in the run equilibrium is denoted by W^{run}. In the run equilibrium, both depositors, regardless of their types, arrive in period 1 and a patient depositor chooses to withdraw if and only if he is the first in the queue. If there is one impatient depositor, the probability of the patient depositor being the first in the queue is $\frac{1}{2}$ and, in this case, there is misallocation of liquidity since the impatient depositor is unable to make his withdrawal. However, if the patient depositor is second in the queue or both depositors are patient, there is no misallocation of liquidity even in the run equilibrium.

As we show in Section 3, depending on the parameters, the ICC might bind for $\gamma \in [\frac{1}{2y}, \gamma^{\text{run-proof}}]$. To analyze the ICC, let $D(\gamma)$ denote the difference between the left-hand side and the right-hand side of Inequality (7). Let Δ denote the difference between asset returns, that is,

$$\Delta = R_A - R_B > 0.$$

The next lemma shows that, if the difference in asset returns is small, a patient depositor's incentive to arrive in period 2 strictly increases with the liquid asset investment.

Lemma 2 *If $\Delta < R_B$, $D(\gamma)$ is strictly increasing in γ.*

For any $\gamma \in [\frac{1}{2y}, \gamma^{\text{run-proof}}]$, complete smoothing of left-over liquidity is restricted by the non-negativity constraint. Hence, if a depositor withdraws in period 1, the remaining depositor receives in period 2 all the liquid asset left in the GSB including its return, namely $(2\gamma y - 1)R_B$. Therefore, with more liquidity, a patient depositor can receive more from the GSB in period 2. However, it is costly to have more liquidity since the remaining depositor's total left-over consumption also includes the illiquid asset and its return, $(1 - \gamma)yR_A$. Hence, if the difference in the asset returns is small (i.e. $\Delta < R_B$), a patient depositor's incentive to arrive in period 2 strictly increases with the bank liquidity.

The condition $\Delta < R_B$ is equivalent to $R_A < 2R_B$, which seems to be a reasonable assumption. We focus on the case of $R_B < R_A < 2R_B$ in the analysis of the optimal contract. We characterize the ICC as a constraint on γ in the following lemma.

Lemma 3 *If $\Delta < R_B$, the ICC is equivalent to $\gamma \geq \gamma^{IC}$. If $D(\frac{1}{2y}) \geq 0$, we have $\gamma^{IC} = \frac{1}{2y}$. If $D(\frac{1}{2y}) < 0$, we have γ^{IC} equal to the level of γ such that $D(\gamma) = 0$.*

From the above analysis, we know that $\gamma^*(s)$ is the level of $\gamma \in [\gamma^{IC}, 1]$ which maximizes $W(\gamma; s)$. For any $\gamma \in [\gamma^{\text{run-proof}}, 1]$, run equilibria can be eliminated and complete consumption smoothing is provided. The best γ is $\gamma^{\text{run-proof}}$. Therefore, we only have to compare $\gamma^{\text{run-proof}}$ with the best γ in the range of $[\gamma^{IC}, \gamma^{\text{run-proof}})$ to find $\gamma^*(s)$. Due to the postponement of the consumption opportunity by the impatient depositor, the expected utility in the run equilibrium is always smaller than the expected utility with $\gamma = \gamma^{\text{run-proof}}$. Therefore, the optimal contract tolerates runs if and only if s is sufficiently small.

Proposition 3 *The optimal contract tolerates runs if and only if the run probability is smaller than a threshold s_0. If the run probability is larger than that threshold, the optimal contract is run-proof (i.e. $\gamma = \gamma^{\text{run}-\text{proof}}$).*

Now let us analyze how the optimal contract responds as the run probability increases from 0 to a sufficiently small value such that it is still optimal to tolerate the runs.

Proposition 4 *If the ICC constraint does not bind at the NROC, then $\gamma^*(s)$ strictly increases with s when s is sufficiently small. If the ICC binds at the NROC, we have $\gamma^*(s) = \gamma^{IC}$ when s is sufficiently small.*

Three numerical examples follow. In the first example, banks are redundant.

Example 1 Let $\overline{u} = 3$, $\beta = 0.5$, $q = 0.5$, $y = 1.1$, $R_A = 1.5$, $R_B = 1.3$, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, where $\theta = 2$. For the unified system, we have $W^{\text{non-run}} = 4.173077$ when $\gamma = 0$ and $W^{\text{non-run}} = 3.681818$ when $\gamma = \frac{1}{2y} = 0.454545$. Therefore $\gamma = 0$ for the NROC.

In the next example, the ICC does not bind for the GSB; for small s, $\gamma^*(s)$ is strictly increasing.

Example 2 The parameters are the same as Example 1, except that $\overline{u} = 10$. For the unified system, we have $W^{\text{non-run}} = 16.423077$ when $\gamma = 0$ and $W^{\text{non-run}} = 17.681818$ when $\gamma = \frac{1}{2y} = 0.454545$. Therefore $\gamma = \frac{1}{2y} = 0.454545$ for the NROC. For the GSB, we have $\gamma^{IC} = 0.750154$ and $\gamma^{\text{run-proof}} = 0.804196$. If $s > s_0 = 0.017708$, the optimal contract is run-proof (i.e. $\gamma^*(s) = \gamma^{\text{run-proof}}$). If $s < s_0$, the optimal contract tolerates runs and $\gamma^*(s)$ is strictly increasing in s. If $s = s_0$, the run-tolerating contract and the run-proof contract would deliver the same *ex ante* welfare. In Figure 1, we plot $\gamma^*(s)$. The jump of γ to $\gamma^{\text{run-proof}}$ at $s = s_0$ dwarfs the increase in $\gamma^*(s)$ below s_0. To show more clearly that $\gamma^*(s)$ does strictly increase in s when $s < s_0$, the plot is separated into two panels.

In the next example, $\gamma^*(s)$ is a step function.

Example 3 The parameters are the same as in Example 2 except that $\theta = 0.9$. For the unified system, we have $W^{\text{non-run}} = 16.656728$ when $\gamma = 0$ and $W^{\text{non-run}} = 18.544118$ when $\gamma = \frac{1}{2y} = 0.454545$. Therefore $\gamma = \frac{1}{2y} = 0.454545$ for the NROC. For the GSB, the ICC binds and we have $\gamma^{IC} = \widehat{\gamma} = 0.741409$. Since $\gamma^{\text{run-proof}}$ only depends on y and R_B, we have the same $\gamma^{\text{run-proof}}$ as

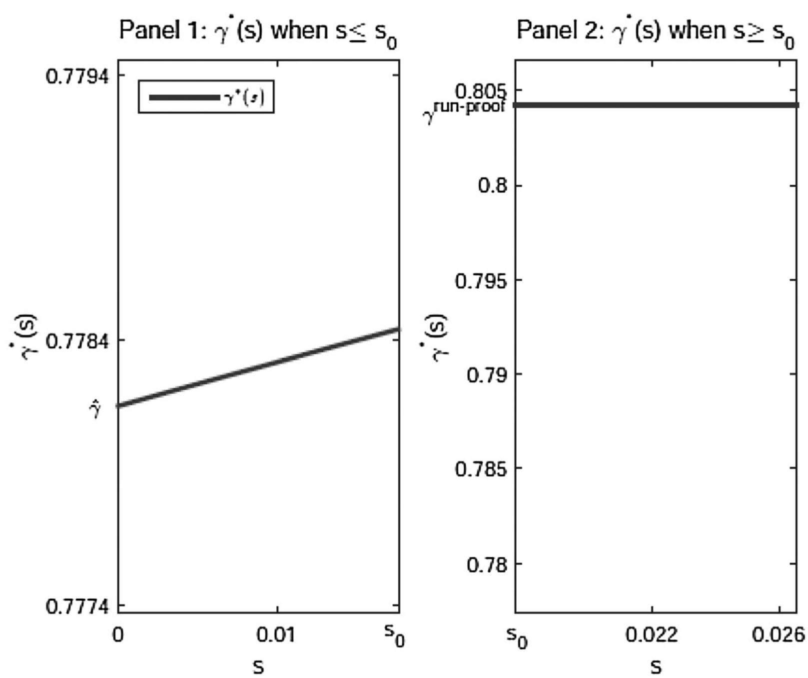

Figure 1 $\gamma^*(s)$ for Example 2

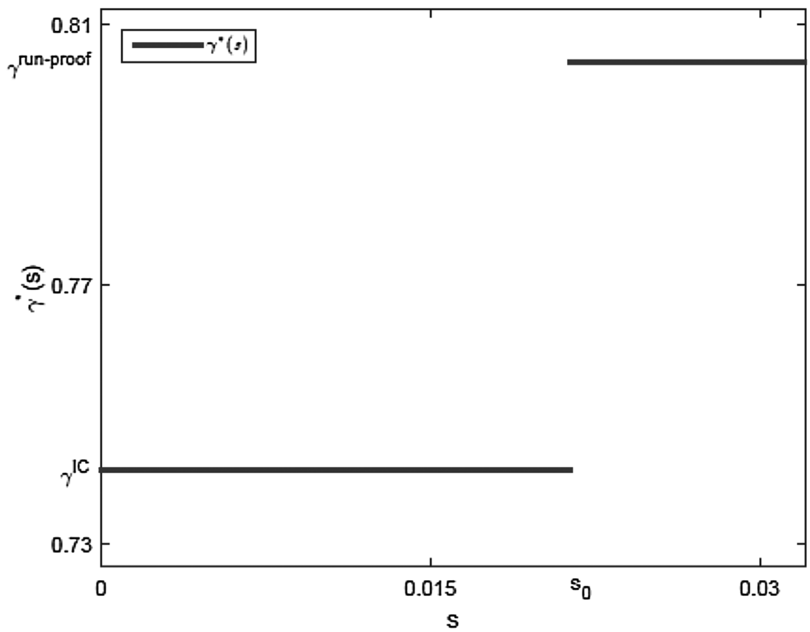

Figure 2 $\gamma^*(s)$ for Example 3

in Example 2. If $s > s_0 = 0.021389$, the optimal contract is run-proof (i.e. $\gamma^*(s) = \gamma^{\text{run-proof}}$). If $s < s_0$, the optimal contract tolerates runs and is locally constant ($\gamma^*(s) = \gamma^{IC}$). In Figure 2, we plot the step function $\gamma^*(s)$.

5 Summary and concluding remarks

In the separated financial system, the GSB has access only to the liquid asset. Therefore, to eliminate runs the GSB would be more liquid than is optimal. Liquidity is costly to the depositor (and the economy) because of the difference in the returns on the two assets. The cost of a run-proof contact cannot be justified unless the run probability is sufficiently high. On the other hand, there is no portfolio restriction for the consolidated bank. It is able to use the illiquid asset to provide incentive for the patient depositor to wait. It is immune to panic-based runs.

We have focused on the GSB where sunspot-driven runs can occur. The optimal banking contract is completely determined by $\gamma^*(s)$, the fraction of resources deposited in the GSB as a function of the exogenous run probability s. For small s, consumers tolerate runs. How does $\gamma^*(s)$ vary with small changes in s? In some cases, $\gamma^*(s)$ is invariant to small changes in s. In these cases, the ICC is binding. Other cases are more intuitive: the fraction $\gamma^*(s)$ increases with small increases in s. That is, depositors respond to increased risk by choosing to be more conservative (i.e. to be more liquid). In these intuitive cases, the ICC is not binding.

If the ICC does not bind, then for small, positive s the sunspot allocation in the pre-deposit game is not a mere randomization over outcomes from the post-deposit game. This places this result in the general sunspot equilibrium literature, from which we know that sunspot equilibria are sometimes mere randomizations over equilibria from the corresponding certainty economy; the

GSB with binding ICC is analogous. Other times, sunspot equilibria are not randomizations over the corresponding certainty equilibria. Indeed, they are entirely different animals; the GSB with non-binding ICC is like one of these different sunspot animals.[8]

Appendix: Proofs

Proof of Proposition 1

For $\gamma = 0$,

$$W^{\text{non-run}} = q(\overline{u} + \beta\overline{u}) + (1-q)(2\overline{u}) + 2u[yR_A - 1].$$

For $\gamma = \frac{1}{2y}$,

$$W^{\text{non-run}} = 2\overline{u} + q2u\left(yR_A - \frac{R_A + 1}{2}\right) + (1-q)2u\left(yR_A - \frac{R_A - R_B}{2} - 1\right).$$

Therefore, if $\overline{u} > \overline{u}_0$, the NROC implies that $\gamma = \frac{1}{2y}$. If $\overline{u} \leq \overline{u}_0$, the NROC implies that $\gamma = 0$.

Proof of Proposition 2

It can be shown that

$$\begin{aligned}\left(\frac{\partial W^{\text{non-run}}}{\partial \gamma}\right)_{\gamma=\gamma^{\text{run-proof}}} &= -2yqu'\left[yR_A - \frac{R_A}{2} - \frac{R_A}{2R_B}\right](R_A - R_B)\\ &\quad -2y(1-q)u'\left[yR_A - \frac{R_A}{2} - \frac{R_A}{2R_B} + \frac{R_B - 1}{2}\right](R_A - R_B)\\ &< 0.\end{aligned}$$

Hence we have $\widehat{\gamma} < \gamma^{\text{run-proof}}$.

Proof of Lemma 2

It can be shown that

$$\begin{aligned}D'(\gamma) = \quad & y\left\{\tfrac{(1-3q/4)}{q/2+(1-q)}u'[(1-\gamma)yR_A]R_A\right.\\ &+\tfrac{q/4}{q/2+(1-q)}u'[(1-\gamma)yR_A + (2\gamma y - 1)R_B - 1](2R_B - R_A)\\ &\left.-\tfrac{(1-q)}{q/2+(1-q)}u'[(1-\gamma)yR_A + \gamma yR_B - 1](R_A - R_B)\right\}.\end{aligned}$$

Therefore, if $\Delta < R_B$, we have $(2R_B - R_A) > 0$. Since

$$u'[(1-\gamma)yR_A] > u'[(1-\gamma)yR_A + \gamma yR_B - 1]$$

8 See Shell (2008). The original sunspots paper (Shell 1977) was set in a dynamic overlapping-generations dynamic model. The *International Journal of Economic Theory* has provided an excellent home for work on economic dynamics, including sunspot dynamics. See, for example, Dufourt *et al*. (2016).

for any $\gamma \in [\frac{1}{2y}, \gamma^{\text{run-proof}}]$, we have $D'(\gamma) > 0$.

Proof of Lemma 3

If $D(\frac{1}{2y}) \geq 0$, we have $D(\gamma) > 0$ for any $\gamma \in [\frac{1}{2y}, \gamma^{\text{run-proof}}]$ since $D(\gamma)$ is strictly increasing. Since $D(\gamma^{\text{run-proof}})$ is always positive, if $D(\frac{1}{2y}) < 0$ there must be a unique level of $\gamma \in [\frac{1}{2y}, \gamma^{\text{run-proof}}]$ such that $D(\gamma) = 0$. Therefore, the ICC is equivalent to $\gamma \geq \gamma^{IC}$.

Proof of Proposition 4

If the ICC does not bind at NROC, we have $(\frac{\partial W^{\text{non-run}}}{\partial \gamma})_{\gamma=\gamma^{IC}} \geq 0$. Then we have $(\frac{\partial W^{\text{run}}}{\partial \gamma})_{\gamma=\gamma^{IC}} > 0$. This is because

$$\begin{aligned}\frac{\partial W^{\text{non-run}}}{\partial \gamma} &= -yqu'[(1-\gamma)yR_A]R_A \\ &\quad + yqu'[(1-\gamma)yR_A + (2\gamma y - 1)R_B - 1](2R_B - R_A) \\ &\quad - 2y(1-q)u'[(1-\gamma)yR_A + \gamma y R_B - 1](R_A - R_B)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial W^{\text{run}}}{\partial \gamma} &= -yu'[(1-\gamma)yR_A]R_A \\ &\quad + yu'[(1-\gamma)yR_A + (2\gamma y - 1)R_B - 1](2R_B - R_A).\end{aligned}$$

We see that if $\frac{\partial W^{\text{non-run}}}{\partial \gamma} \geq 0$, we have $\frac{\partial W^{\text{run}}}{\partial \gamma} > \frac{\partial W^{\text{non-run}}}{\partial \gamma} > 0$ and therefore

$$\begin{aligned}\left(\frac{\partial W(\gamma; s)}{\partial \gamma}\right)_{\gamma=\gamma^*(s)} &= s\left(\frac{\partial W^{\text{run}}}{\partial \gamma}\right)_{\gamma=\gamma^*(s)} + (1-s)\left(\frac{\partial W^{\text{non-run}}}{\partial \gamma}\right)_{\gamma=\gamma^*(s)} \\ &= 0\end{aligned}$$

for s sufficiently small. According to the implicit function theorem, we have

$$\begin{aligned}\frac{\partial \gamma^*(s)}{\partial s} &= \left[\frac{\frac{\partial W^{\text{non-run}}}{\partial \gamma} - \frac{\partial W^{\text{run}}}{\partial \gamma}}{s\frac{\partial^2 W^{\text{run}}}{\partial \gamma^2} + (1-s)\frac{\partial^2 W^{\text{non-run}}}{\partial \gamma^2}}\right]_{\gamma=\gamma^*(s)} \\ &> 0.\end{aligned}$$

If the ICC does not bind at the NROC, we have $(\frac{\partial W^{\text{non-run}}}{\partial \gamma})_{\gamma=\gamma^{IC}} < 0$. Then for s sufficiently small,

$$\begin{aligned}\left(\frac{\partial W(\gamma; s)}{\partial \gamma}\right)_{\gamma=\gamma^{IC}} &= s\left(\frac{\partial W^{\text{run}}}{\partial \gamma}\right)_{\gamma=\gamma^{IC}} + (1-s)\left(\frac{\partial W^{\text{non-run}}}{\partial \gamma}\right)_{\gamma=\gamma^{IC}} \\ &< 0.\end{aligned}$$

Therefore, $\gamma^*(s) = \gamma^{IC}$ for s is small.

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